Math 461 Final Exam May 22, 2000

1. (20) Let A be any 3×3 matrix with eigenvalues 1, -2, and 3. Answer the following, giving adequate reasons.
 a) What are the eigenvalues of A^T?
 b) Does A^{-1} necessarily exist? If it does, what are the eigenvalues of A^{-1}?
 c) What are the eigenvalues of A^2?
 d) Is A necessarily diagonalizable?
 e) Is A necessarily orthogonally diagonalizable?

2. (20) Let u_1, u_2, u_3, u_4 be an orthonormal set of vectors in a four dimensional inner product space V.
 a) What is $||2u_1 + 3u_2 - u_3||$?
 b) Show that $u_1 - 2u_3 + u_4$ and $2u_1 - u_2 + u_3$ are orthogonal.
 c) Is u_1, u_2, u_3, u_4 a linearly independent list of vectors?
 d) Is u_1, u_2, u_3, u_4 a basis of V?

3. (20) Find an orthogonal basis for P_2 with inner product $< p, q >= \int_{-1}^{1} p(t)q(t)\,dt$. Next find an orthonormal basis for P_2 (with the same inner product).

4. (15) Let $T\colon P_2 \to \mathbb{R}^3$ be the linear transformation $T(p) = [p(0) \quad p(1) - p(-1) \quad p(2)]^T$. Find the matrix for T relative to the basis $\{t^2, t, 1\}$ of P_2 and $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.

5. (26) Let x_1 and x_2 be two different least squares solutions to $A\hat{x} = b$ for a 7×3 matrix A. (So $x_1 \neq x_2$). Could the rank of A be 3? What can you say about $Ax_1 - Ax_2$? What can you say about $Ax_1 - b$? Give sufficient reasons for your answers. Find the least squares solution to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

6. (25) Find an orthogonal coordinate change matrix P which transforms the quadratic form $x^2 + xy + y^2 + z^2$ into one with no cross product terms.

7. (10) Suppose A is a real 5×5 matrix whose only eigenvalues are 1 and $2 \pm 3i$. (Hint: This problem has to do with block diagonal form and real Jordan canonical form.)
 a) Give an example of such a real A so that $\mathbb{C}^5$ has a basis of (complex) eigenvectors of A.
 b) Give an example of such a real A so that $\mathbb{C}^5$ does not a basis of (complex) eigenvectors of A.

8. (10) Give an explicit example of the singular value decomposition of a 3×4 matrix with rank 2. (You need not multiply it out).

9. (54) True (always true), False (always false), Maybe (sometimes true and sometimes false, depending on A, S, etc.) or short answer.

a) If the characteristic polynomial of A is $\lambda(\lambda-3)(\lambda+3)(\lambda-10)$ then A is diagonalizable.

b) Two eigenvectors of a symmetric matrix are ________ if they correspond to different eigenvalues.

c) If A is invertible, then A is row equivalent to the identity matrix.

d) If A is similar to B and A is singular, then B is singular.

e) If $L\colon \mathbb{R}^9 \to \mathbb{R}^9$ is a linear transformation and the kernel of L is $\{0\}$, then the range of L is all of $\mathbb{R}^9$.

f) If x and y are two vectors with the same length in an inner product space, then $(x-y) \perp (x+y)$.

g) Any square matrix A can be written as $A = SJS^{-1}$ where J is in Jordan canonical form.

h) For A, B, and C matrices, if $AB = AC$ then $B = C$.

i) The eigenvalues of a symmetric matrix are all real.

j) $\det(BA) = \det(AB)$.

k) If A is an $n \times n$ matrix, then A is diagonalizable if $\mathbb{R}^n$ has a basis of eigenvectors of A.

l) If A has eigenvalues 1,2 and 3 then $A + 3I$ has eigenvalues ________.

m) $Col(A)^{\perp} =$ ________.

n) If S is a subspace of an inner product space V, then define $S^{\perp}$.

o) If A is an $m \times n$ matrix with rank n, then the matrix of orthogonal projection to $Col(A)$ is ________.

p) If n vectors span an m dimensional space, then

  i) $n \geq m$
  ii) $n \leq m$
  iii) $n = m$
  iv) You could sometimes have $n \geq m$ and sometimes have $n < m$, depending on the vectors.

q) If there are n linearly independent vectors in an m dimensional space, then

  i) $n \geq m$
  ii) $n \leq m$
  iii) $n = m$
  iv) You could sometimes have $n \geq m$ and sometimes have $n < m$, depending on the vectors.

r) $Ax \cdot y = x \cdot A^T y$.