Math 461 Final Exam May 22, 2000

1. (25) Let A be a 3×3 symmetric matrix with rank 2 and suppose $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. What are all the eigenvalues of A? Is A diagonalizable? If possible, find an orthogonal matrix P and a diagonal matrix D so that $A = PDP^{-1}$.

2. (25) Let Q be an orthogonal matrix.
 a) What is $Q^T Q$?
 b) Show that $\det(Q) = \pm 1$. (Hint: What is $\det(Q^T)$?)
 c) If $x \neq 0$, what can you say about $||Qx||/||x||$?
 d) Show that all real eigenvalues of Q must be either 1 or -1.

3. (20) Find an orthogonal basis for P_2 with inner product $< p, q >= \int_{-1}^{1} p(t)q(t)\, dt$. Next find an orthonormal basis for P_2 (with the same inner product).

4. (15) Let $T\colon P_2 \to \mathbb{R}^3$ be the linear transformation $T(p) = [p(0) \quad p(1) - p(-1) \quad p(2)]^T$. Find the matrix for T relative to the basis $\{t^2, t, 1\}$ of P_2 and $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.

5. (10) Find a least squares solution to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Are there any other least squares solutions?

6. (25) Find an orthogonal coordinate change matrix P which transforms the quadratic form $y^2 - 2\sqrt{2}xy$ into one with no cross product terms.

7. (10) Give an example of a matrix in Jordan canonical form which is not diagonalizable.

8. (10) Give an explicit example of the singular value decomposition of a 3×2 matrix with rank 1. (You need not multiply it out).

9. (60) True (always true), False (always false), Maybe (sometimes true and sometimes false, depending on A, S, etc.) or short answer.
 a) If the characteristic polynomial of A is $\lambda(\lambda - 3)^3(\lambda - 10)$ then A is diagonalizable.

b) Two eigenvectors of a symmetric matrix are ________ if they correspond to different eigenvalues.

c) If A is invertible, then A is row equivalent to the identity matrix.

d) If A is similar to B and A is singular, then B is singular.

e) If $L\colon \mathbb{R}^9 \to \mathbb{R}^9$ is a linear transformation and the kernel of L is $\{0\}$, then the range of L is all of $\mathbb{R}^9$.

f) If x and y are two vectors with the same length in an inner product space, then $x - y$ is orthogonal to $x + y$.

g) Any square matrix A can be written as $A = SJS^{-1}$ where J is in Jordan canonical form.

h) For A, B, and C matrices, if $AB = AC$ then $B = C$.

i) The eigenvalues of a symmetric matrix are all real.

j) $\det(BA) = \det(AB)$.

k) If A is an $n \times n$ matrix, then A is diagonalizable if $\mathbb{R}^n$ has a basis of eigenvectors of A.

l) If A has eigenvalues 1,2 and 3 then $A + 3I$ has eigenvalues ________.

m) $Col(A)^{\perp} =$ ________.

n) If A and B are nonsingular then $(AB)^{-1} = A^{-1}B^{-1}$.

o) If A is an $m \times n$ matrix with rank n, then the matrix of orthogonal projection to $Col(A)$ is ________.

p) If n vectors span an m dimensional space, then
   i) $n \geq m$
   ii) $n \leq m$
   iii) $n = m$
   iv) You could sometimes have $n \geq m$ and sometimes have $n < m$, depending on the vectors.

q) If there are n linearly independent vectors in an m dimensional space, then
   i) $n \geq m$
   ii) $n \leq m$
   iii) $n = m$
   iv) You could sometimes have $n \geq m$ and sometimes have $n < m$, depending on the vectors.

r) $Ax \cdot y = x \cdot A^T y$.

s) For any matrix A, the matrix $A^T A$ is symmetric.

t) For any matrix A, all eigenvalues of $A^T A$ are nonnegative.