Math 461 Section 0601 Final Exam May 17, 2004

1. (45) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 \\ 2 & 4 & 4 & 2 & 4 \\ 2 & 3 & 4 & 0 & 0 \\ 1 & 3 & 2 & 2 & 5 \end{bmatrix}$$

a) Find the reduced echelon form of A.

b) Find the rank of A.

c) Find a basis for the column space of A.

d) Find a basis for the Null space of A.

e) Find all solutions to $A\mathbf{x} = [1 \quad 0 \quad 1 \quad 0]^T$.

f) Find all solutions to $A\mathbf{x} = [2 \quad 2 \quad 1 \quad 0]^T$.

2. (30) For each of the following matrices:

+ Find its eigenvalues and an eigenvector for each eigenvalue.

+ If possible, find a (possibly complex) matrix P and a diagonal matrix D so that the given matrix equals PDP^{-1}. If possible, P should be orthogonal.

+ If possible, find a real matrix Q so that the given matrix is QCQ^{-1} where C is of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

+ Find a formula for $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}^k$ as a product of at most 3 matrices.

$$\text{a) } \begin{bmatrix} 5 & 8 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 4 \end{bmatrix} \qquad \text{b) } \begin{bmatrix} -4 & 5 \\ -5 & 4 \end{bmatrix} \qquad \text{c) } \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

3. (15) A matrix A has singular value decomposition

$$A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \end{bmatrix}^T$$

a) Find an orthonormal basis for the column space of A.

b) Find an orthonormal basis for the Null space of A.

4. (25) Find an orthogonal basis for Span$\{1, t, t^2\}$ in $C[0, 2]$ using the inner product $\langle f,\ g\rangle = \int_0^2 f(t)g(t)\,dt$. Suppose $f(t)$ is a function in $C[0, 2]$. Find the projection of f to Span$\{1, t, t^2\}$ if $\langle 1,\ f\rangle = 4$, $\langle t,\ f\rangle = 32/5$, $\langle t^2,\ f\rangle = 32/3$, and $\langle t^3,\ f\rangle = 128/7$.

5. (25) Determine whether each of the following subsets of $\mathbb{R}^5$ are subspaces and find a basis and dimension if they are.

a) S_1 is the set of $[x_1, x_2, x_3, x_4, x_5]^T$ so that $x_1 + x_2 + x_3 + x_4 + x_5 = 1$.

b) S_2 is the set of $[x_1, x_2, x_3, x_4, x_5]^T$ so that $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ and $x_1 + x_2 + x_3 = x_4 + x_5$.

c) S_3 is Span$\{[1, 2, 3, 4, 5]^T, [1, 1, 1, 1, 1]^T, [0, 1, 2, 3, 4]^T\}$.

6. (60) True (always true), False (always false), Maybe (sometimes true and sometimes false, depending on A, S, etc.) or short answer. A and B are 8×8 matrices, C is a 4×8 matrix, and S is a four dimensional subspace of a seven dimensional real vector space V with an inner product.

a) The eigenvalues of a Hermitian matrix are all real.

b) Two eigenvectors of a symmetric matrix are orthogonal if they correspond to different eigenvalues.

c) Using the usual Hermitian inner product in $\mathbb{C}^3$ the length of $[1 + i, 2 - i, 3]^T$ is $\sqrt{(1 + i)^2 + (2 - i)^2 + 9}$.

d) If the characteristic polynomial of A has a repeated root, then A is not diagonalizable.

e) If C has rank 3 then the null space of C has dimension 1.

f) There is a set of 6 linearly independent vectors in P_4 .

g) If $[u_1, \ldots, u_7]$ is an orthonormal basis for V, then

$$\langle 2u_1 + 3u_2 - u_4 + u_6,\ u_1 - 2u_2 + u_4 + u_7\rangle = -5$$

h) V has an orthonormal basis.

i) Any orthogonal set in V is linearly independent.

j) $(S^\perp)^\perp =$ ________.

k) $(\text{Nul}A)^\perp =$ ________.

l) $(AB)^T =$ ________.