Math 461 Section 0501 Final Exam May 18, 2004

1. (60) Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

a) Find the reduced echelon form of A.

b) Find the rank of A.

c) Find an orthogonal basis for the column space of A.

d) Find an orthonormal basis for the Null space of A.

e) Find all solutions to $A\mathbf{x} = [8 \quad 0 \quad 6 \quad -2]^T$.

f) Find all solutions to $A\mathbf{x} = [2 \quad 2 \quad 1 \quad 0]^T$.

g) Find the projection of $[1, 0, 0, 0]^T$ to the column space of A.

2. (15) Find the third order Fourier approximation to the square wave function $f(t) = 1$ for $0 \le t < \pi$ and $f(t) = 0$ for $\pi \le t < 2\pi$. In other words, find the closest function to f in $\text{Span}\{1, \sin t, \sin 2t, \sin 3t, \cos t, \cos 2t, \cos 3t\}$ using the inner product $\langle f,\ g\rangle = \int_0^{2\pi} f(t)g(t)\,dt$.

3. (20) Let $Q(\mathbf{x}) = 3x_1^2 + x_2^2 + x_3^2 + 4x_2x_3$ be a quadratic form. Find an orthogonal matrix P so that the change of variables $\mathbf{x} = P\mathbf{y}$ transforms Q to a quadratic form with no cross-product terms. Determine whether or not Q is positive definite, negative definite,or indefinite.

4. (20) Let $T\colon P_3 \to P_2$ be the linear transformation $T(p) = p'(t) + p(0)$. Find the matrix for T relative to the basis $\{1,\ t-1,\ t-t^2,\ t^3\}$ of P_3 and the basis $\{t^2, 1, t\}$ of P_2.

5. (25) Determine whether each of the following subsets of P_3 are subspaces and find a basis and its dimension if it is.

a) S_1 is the set of $p(t)$ in P_3 so that $p(1) = 0$.

b) S_2 is the set of $p(t)$ in P_3 so that $p(0) = 1$.

c) S_3 is $\text{Span}\{1, t-1, t^3 - t^2, t^3 - t^2 - t\}$.

d) S_4 is the set of $p(t)$ in P_3 so that $p(0) = p(1) = 0$.

6. (60) True (always true), False (always false), Maybe (sometimes true and sometimes false, depending on A, S, etc.) or short answer. A and B are real 8×8 matrices, C is a complex 4×8 matrix, and S is a four dimensional subspace of a seven dimensional real vector space V with an inner product.

a) Two eigenvectors of a symmetric matrix are orthogonal if they correspond to different eigenvalues.

b) Using the usual Hermitian inner product in $\mathbb{C}^3$ the length of $[1+i, 2-i, 3]^T$ is $\sqrt{(1+i)^2 + (2-i)^2 + 9}$.

c) If the characteristic polynomial of A has a repeated root, then A is not diagonalizable.

d) CC^* is diagonalizable.

e) There are 6 vectors in P_4 which span P_4.

f) If $\{u_1, \ldots, u_7\}$ is an orthonormal basis of V, then

$$\{u_1 + 2u_2 - 5u_3 + u_4,\ 2u_1 - u_2,\ u_1 + 2u_2 + u_3\}$$

is an orthogonal set.

g) If A has no real eigenvalues, then A is nonsingular.

h) V has an orthonormal basis.

i) Any orthogonal set in V is linearly independent.

j) $\dim S^{\perp} =$ _________.

k) If C has rank 3 then $\dim Nul(C) =$ _________.

l) If A and B are invertible, then $(AB)^{-1} =$ _________.