Math 461 Test # 3, 0501

1. (25) For each of the following matrices:
 + Find its eigenvalues and an eigenvector for each eigenvalue.
 + If possible, find a (possibly complex) matrix P and a diagonal matrix D so that the given matrix equals PDP^{-1}.
 + If possible, find a real matrix Q so that the given matrix is QCQ^{-1} where C is of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$$\text{a)} \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} -6 & -15 \\ 3 & 6 \end{bmatrix}$$

Answer: The characteristic polynomial of a) is $(4-\lambda)(-2-\lambda) - 4(-2) = \lambda^2 - 2\lambda$ *so the eigenvalues are 0 and 2. For* $\lambda = 2$ *the eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix}$ *so an eigenvector is* $(2,-1)^T$. *For* $\lambda = 0$ *the eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix}$ *so an eigenvector is* $(1,-1)^T$. *So the matrix is diagonalizable and equals* PDP^{-1} *where* $P = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ *and* $D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. *It cannot be* QCQ^{-1} *since its eigenvalues would then be the eigenvalues of* C *which are* $a \pm b\sqrt{-1}$. *The characteristic polynomial of b) is* $(1-\lambda)(1-\lambda) - 2(0) = (\lambda-1)^2$ *so the only eigenvalue is 1. The eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ *so an eigenvector is* $(1,0)^T$. *It is not diagonalizable since the eigenvalue 1 has multiplicity 2 but its eigenspace is only one dimensional. It is not* QCQ^{-1} *since the eigenvalues of* C *are* $a \pm b\sqrt{-1}$ *which would mean* $a = 1$ *and* $b = 0$ *so* C *would be diagonal, but the matrix is not diagonalizable. The characteristic polynomial of c) is* $(-6-\lambda)(6-\lambda) - 3(-15) = \lambda^2 + 9$ *so the eigenvalues are* $\pm 3\sqrt{-1}$. *For* $\lambda = 3\sqrt{-1}$ *the eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} -6-3\sqrt{-1} & -15 \\ 3 & 6-3\sqrt{-1} \end{bmatrix}$ *so an eigenvector is* $(6-3\sqrt{-1}, -3)^T$ *or more simply* $(-2+\sqrt{-1}, 1)^T$. *For* $\lambda = -3\sqrt{-1}$ *an eigenvector is the complex conjugate,* $(-2-\sqrt{-1}, 1)^T$. *So the matrix is* PDP^{-1} *with* $P = \begin{bmatrix} -2+\sqrt{-1} & -2-\sqrt{-1} \\ 1 & 1 \end{bmatrix}$ *and* $D = \begin{bmatrix} 3\sqrt{-1} & 0 \\ 0 & -3\sqrt{-1} \end{bmatrix}$. *It is also* QCQ^{-1} *where* $Q = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$ *has columns which are the real and imaginary parts of the eigenvectors.*

2. (20) A 5×5 matrix A has three eigenvalues 1, 3, and 6. The eigenspace of A corresponding to $\lambda = 1$ is three dimensional.
 a) What is the characteristic polynomial of A?

 Answer: We know that $\lambda = 1$ *has multiplicity* ≥ 3. *So since it has degree 5 the only possibility is* $-(\lambda-1)^3(\lambda-3)(\lambda-6)$. *(The minus sign comes because the coefficient* *of λ^n in Lay's version of the characteristic polynomial is $(-1)^n$ for an $n \times n$ matrix. I did not take off points if you got the sign wrong.)*

b) Must A be diagonalizable? Why or why not?

Answer: Since the eigenspace dimensions all equal the multipicities we know A must be diagonalizable.

c) Must A be invertible? Why or why not?

Answer: Since 0 is not an eigenvalue, the null space of A is trivial, so A is invertible.

d) Find the ranks of the matrices A, $A - I_5$ and $A - 3I_5$.

Answer: Since A is invertible, $\text{rank}(A) = 5$. The eigenspace for $\lambda = 1$ is the null space of $A - I_5$ so $\text{rank}(A - I_5) = 5 - \dim(Null(A - I_5)) = 2$. Likewise, $\text{rank}(A - 3I_5) = 5 - \dim(Null(A - 3I_5)) = 4$ since the eigenspace for $\lambda = 3$ has dimension at most the multiplicity 1.

3. (15) Suppose A is a 3×3 matrix so that $A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$, and so that $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is in the null space of A.

a) What are the eigenvalues of A?

Answer: Since $A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ we know 2 is an eigenvalue, since $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ we know -1 is an eigenvalue, and since $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ we know 0 is an eigenvalue. Since A is 3×3 it has at most 3 eigenvalues so the eigenvalues are 2, -1, and 0.

b) Determine A. (You may leave your answer as a product of matrices and their inverses.)

Answer: We know from part a) that $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c) Determine A^k for any integer $k > 0$. (You may leave your answer as a product of matrices and their inverses.)

Answer: We know $A^k = PD^kP^{-1}$ where P and D are as in part b), so

$$A^k = P \begin{bmatrix} 2^k & 0 & 0 \\ 0 & (-1)^k & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}$$

4. (10) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation which takes a polynomial $p(t)$ to $tp'(t)$. Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B} = \{1, t, t^2\}$.

Answer: $T(1) = t \cdot 1' = 0$, $T(t) = t \cdot t' = t$, and $T(t^2) = t \cdot (t^2)' = 2t^2$. So

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

5. (15) Suppose $B = P^{-1}AP$.

a) Then by definition, A is __________ to B.

Answer: similar

b) Show that if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B with eigenvalue λ.

Answer: We are given that $x \neq 0$ and $Ax = \lambda x$. If $P^{-1}x = 0$ then $x = PP^{-1}x = P(0) = 0$, a contradiction, so $P^{-1}x \neq 0$. Now $BP^{-1}x = P^{-1}APP^{-1}x = P^{-1}Ax = P^{-1}(\lambda x) = \lambda P^{-1}x$, so $P^{-1}x$ is an eigenvector of B with eigenvalue λ.

c) Show that $\det A = \det B$.

Answer:

$$\det(B) = \det(P^{-1}AP) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)(1/\det(P)) = \det(A)$$

6. (15) Same as a problem on Test 2.