Math 461 Test # 3, 0601

1. (25) For each of the following matrices:
   + Find its eigenvalues and an eigenvector for each eigenvalue.
   + If possible, find a (possibly complex) matrix P and a diagonal matrix D so that the given matrix equals PDP^{-1}.
   + If possible, find a real matrix Q so that the given matrix is QCQ^{-1} where C is of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$$\text{a)} \begin{bmatrix} 5 & 8 \\ -2 & -3 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} -6 & -10 \\ 4 & 6 \end{bmatrix}$$

Answer: *The characteristic polynomial of a) is* $(5-\lambda)(-3-\lambda) - 8(-2) = (\lambda-1)^2$ *so the only eigenvalue is* 1. *The eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} 4 & 8 \\ -2 & -4 \end{bmatrix}$ *so an eigenvector is* $(2,-1)^T$. *It is not diagonalizable since the eigenvalue 1 has multiplicity 2 but its eigenspace is only one dimensional. It is not* QCQ^{-1} *since its eigenvalues would then be the eigenvalues of* C *which are* $a \pm b\sqrt{-1}$. *This would mean* $a = 1$ *and* $b = 0$ *so* C *would be diagonal, but the matrix is not diagonalizable. The characteristic polynomial of b) is* $(1-\lambda)(-1-\lambda) - 4(0) = (\lambda-1)(\lambda+1)$ *so the eigenvalues are* 1 *and* -1. *For* $\lambda = 1$ *the eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix}$ *so an eigenvector is* $(1,0)^T$. *For* $\lambda = -1$ *the eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$ *so an eigenvector is* $(2,-1)^T$. *So the matrix is diagonalizable and equals* PDP^{-1} *where* $P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ *and* $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. *It cannot be* QCQ^{-1} *since the eigenvalues of* C *are* $a \pm b\sqrt{-1}$. *The characteristic polynomial of c) is* $(-6-\lambda)(6-\lambda) - 4(-10) = \lambda^2 + 4$ *so the eigenvalues are* $\pm 2\sqrt{-1}$. *For* $\lambda = 2\sqrt{-1}$ *the eigenvectors are nonzero vectors in the null space of* $\begin{bmatrix} -6-2\sqrt{-1} & -10 \\ 4 & 6-2\sqrt{-1} \end{bmatrix}$ *so an eigenvector is* $(6-2\sqrt{-1}, -4)^T$ *or more simply* $(3-\sqrt{-1}, -2)^T$. *For* $\lambda = -2\sqrt{-1}$ *an eigenvector is the complex conjugate,* $(3+\sqrt{-1}, -2)^T$. *So the matrix is* PDP^{-1} *with* $P = \begin{bmatrix} 3-\sqrt{-1} & 3+\sqrt{-1} \\ -2 & -2 \end{bmatrix}$ *and* $D = \begin{bmatrix} 2\sqrt{-1} & 0 \\ 0 & -2\sqrt{-1} \end{bmatrix}$. *It is also* QCQ^{-1} *where* $Q = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$ *has columns which are the real and imaginary parts of the eigenvectors.*

2. (20) A 5×5 matrix A has three eigenvalues 0, 1, and 3. The eigenspaces of A corresponding to $\lambda = 1$ and $\lambda = 3$ are both two dimensional.
   a) What is the characteristic polynomial of A?

   Answer: *We know that* $\lambda = 1$ *and* 3 *have multiplicity* ≥ 2. *So since it has degree 5 the only possibility is* $-(\lambda-1)^2(\lambda-3)^2\lambda$. *(The minus sign comes because the coefficient* *of λ^n in Lay's version of the characteristic polynomial is $(-1)^n$ for an $n \times n$ matrix. I did not take off points if you got the sign wrong.)*

b) Must A be diagonalizable? Why or why not?

Answer: Since the eigenspace dimensions all equal the multipicities we know A must be diagonalizable.

c) Must A be invertible? Why or why not?

Answer: Since 0 is an eigenvalue, the null space of A is nontrivial, so A is not invertible.

d) Find the ranks of the matrices A, $A - I_5$ and $A - 3I_5$.

Answer: The eigenspace for $\lambda = 0$ is the null space of A so $\text{rank}(A) = 5 - \dim(Null(A)) = 4$ since the eigenspace for $\lambda = 0$ has dimension at most the multiplicity 1. The eigenspace for $\lambda = 1$ is the null space of $A - I_5$ so $\text{rank}(A - I_5) = 5 - \dim(Null(A - I_5)) = 3$. Likewise, $\text{rank}(A - 3I_5) = 5 - \dim(Null(A - 3I_5)) = 3$.

3. (15) Suppose A is a 3×3 matrix so that $A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $A\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$, and so that $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is in the null space of A.

a) What are the eigenvalues of A?

Answer: Since $A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ we know 1 is an eigenvalue, since $A\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$ we know -1 is an eigenvalue, and since $A\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ we know 0 is an eigenvalue. Since A is 3×3 it has at most 3 eigenvalues so the eigenvalues are 1, -1, and 0.

b) Determine A. (You may leave your answer as a product of matrices and their inverses.)

Answer: We know from part a) that $A = PDP^{-1}$ where $P = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$

c) Show that $A = A^{53}$.

Answer: We know $A^{53} = PD^{53}P^{-1}$ where P and D are as in part b), so

$$A^{53} = P\begin{bmatrix}1^{53} & 0 & 0\\0 & (-1)^{53} & 0\\0 & 0 & 0^{53}\end{bmatrix}P^{-1} = P\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}P^{-1} = A$$

4. (10) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation which takes a polynomial $p(t)$ to $tp'(t)$. Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B} = \{1, t, t^2\}$.

Answer: $T(1) = t \cdot 1' = 0$, $T(t) = t \cdot t' = t$, and $T(t^2) = t \cdot (t^2)' = 2t^2$. *So* $[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

5. (15) Suppose $B = P^{-1}AP$.

a) Then by definition, A is __________ to B.

Answer: similar

b) Show that if $\mathbf{x}$ is an eigenvector of A with eigenvalue λ, then $P^{-1}\mathbf{x}$ is an eigenvector of B with eigenvalue λ.

Answer: *We are given that* $x \neq 0$ *and* $Ax = \lambda x$. *If* $P^{-1}x = 0$ *then* $x = PP^{-1}x = P(0) = 0$, *a contradiction, so* $P^{-1}x \neq 0$. *Now* $BP^{-1}x = P^{-1}APP^{-1}x = P^{-1}Ax = P^{-1}(\lambda x) = \lambda P^{-1}x$, *so* $P^{-1}x$ *is an eigenvector of* B *with eigenvalue* λ.

c) Show that $\det A = \det B$.

Answer:

$$\det(B) = \det(P^{-1}AP) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)(1/\det(P)) = \det(A)$$

6. (15) Same as a problem on Test 2.