Math 461 Minitest # 4, 0501

1. (10) Suppose $A = \begin{bmatrix} 1/\sqrt{2} & 1/2 \\ 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$ is the QR decomposition of A.

a) Find an orthonormal basis of the column space of A.

Answer: *Get this from the columns of Q. So a basis is* $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ -1/2 \end{bmatrix} \right\}$. *Note, the matrix Q is* **not** *a basis, it is a matrix, not a basis.*

b) Find the closest point to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the column space of A.

Answer: *This is the projection to the column space. We have an orthonormal basis $\{u_1, u_2\}$ of the column space of A so the projection of e_1 is*

$$\langle u_1,\ e_1\rangle u_1 + \langle u_2,\ e_1\rangle u_2 = 1/\sqrt{2}u_1 + 1/2u_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/(2\sqrt{2}) \\ -1/4 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/(2\sqrt{2}) \\ 1/4 \end{bmatrix}$$

c) Find the least squares solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Answer: *This is the solution to $R\hat{\mathbf{x}} = Q^T e_1$ so $\begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/2 \end{bmatrix}$ so $x_2 = 1/2$, $4x_1 + x_2 = 1/\sqrt{2}$ and $x_1 = (1/\sqrt{2} - 1/2)/4 = (\sqrt{2} - 1)/8$. So the least squares solution is $\hat{\mathbf{x}} = \begin{bmatrix} (\sqrt{2}-1)/8 \\ 1/2 \end{bmatrix}$.*

2. (10) Let $Q(\mathbf{x}) = -3x_2^2 + 3x_3^2 + 12x_1x_2 + 12x_1x_3$.

a) Find a symmetric A so that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

Answer: $\begin{bmatrix} 0 & 6 & 6 \\ 6 & -3 & 0 \\ 6 & 0 & 3 \end{bmatrix}$

b) A has eigenvectors $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ for eigenvalues 9, -9, and 0. Find, if possible, an orthogonal matrix P so that the change of variables $\mathbf{x} = P\mathbf{y}$ transforms Q to a quadratic form with no cross-product terms. If this is not possible, say why not.

Answer: *The columns of P are unit eigenvectors, so it is only necessary to divide the given eigenvectors by their length which is 3 in each case. So* $P = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 1/3 & -2/3 & -2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}$

3. (10) Find an orthogonal basis for Span$\{1, t^2\}$ in $C[-2,2]$ with inner product $\langle f, g\rangle = \int_{-2}^{2} f(t)g(t)\,dt$.

Answer: *By the Gram-Schmidt process we must compute* $t^2 - \frac{\langle 1,\ t^2\rangle}{\langle 1,\ 1\rangle}1$. *We have* $\langle 1,\ t^2\rangle = \int_{-2}^{2} t^2\,dt = t^3/3]_{-2}^{2} = 16/3$ *and* $\langle 1,\ 1\rangle = \int_{-2}^{2} 1\,dt = t]_{-2}^{2} = 4$. *So* $t^2 - \frac{\langle 1,\ t^2\rangle}{\langle 1,\ 1\rangle}1 = t^2 - 4/3$ *and an orthogonal basis is* $\{1,\ t^2 - 4/3\}$.

4. (10) Let $A = \begin{bmatrix} .6 & .8 \\ -.8 & .6 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/2 & 1/2 \end{bmatrix}^T$ be the SVD decomposition of A.

a) Find a unit vector $\mathbf{v}$ so that $||A\mathbf{v}||$ is as large as possible.

Answer: *If* $A = U\Sigma V^T$ *this is the first column of* V *so* $\mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$.

b) What are the eigenvalues of A^TA?

Answer: *They are the squares of the singular values and possibly 0, if the rank of* A *is less than the number of columns of* A. A *has rank 2 since there are two singular values, and* A *has three columns so 0 is an eigenvalue of* A. *So the eigenvalues of* A^TA *are 16, 1, and 0. You could also see this directly since* $V^{-1}A^TAV = V^TA^TAV = \Sigma^T\Sigma$ *which is diagonal with entries 16, 1, and 0. (Recall similar matrices have the same eigenvalues).*

c) What are the singular values of A?

Answer: $\sigma_1 = 4$ *and* $\sigma_2 = 1$.

Do either problem 5 or 6. Problem 6 has a 5 point bonus. Clearly indicate which problem you want graded.

5. (10) Let $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ be an orthonormal set in $\mathbb{R}^4$.

a) Is $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ a linearly independent set?

Answer: *Yes, any orthonormal set is linearly independent.*

b) Find the dot product of $2\mathbf{u_1} + 3\mathbf{u_2}$ with $\mathbf{u_2} - \mathbf{u_3}$.

Answer:

$$(2\mathbf{u_1} + 3\mathbf{u_2}) \cdot (\mathbf{u_2} - \mathbf{u_3}) = 2\mathbf{u_1} \cdot \mathbf{u_2} + 3\mathbf{u_2} \cdot \mathbf{u_2} - 2\mathbf{u_1} \cdot \mathbf{u_3} - 3\mathbf{u_2} \cdot \mathbf{u_3} = 0 + 3 - 0 - 0 = 3$$

6. (15) Let $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ be an orthonormal set in $\mathbb{C}^4$ with the usual Hermitian inner product $\langle \mathbf{u}, \mathbf{v}\rangle = u^*v$.

a) Is $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ a linearly independent set?

Answer: *Yes, any orthonormal set is linearly independent.*

b) Find the inner product $\langle (2+i)\mathbf{u_1} + (3-i)\mathbf{u_2} + \mathbf{u_3},\ 2i\mathbf{u_2} - (1+i)\mathbf{u_3}\rangle$.

Answer:

$$\langle (2+i)\mathbf{u_1} + (3-i)\mathbf{u_2} + \mathbf{u_3},\ 2i\mathbf{u_2} - (1+i)\mathbf{u_3}\rangle$$

$$= (2-i)\langle \mathbf{u_1},\ 2i\mathbf{u_2} - (1+i)\mathbf{u_3}\rangle + (3+i)\langle \mathbf{u_2},\ 2i\mathbf{u_2} - (1+i)\mathbf{u_3}\rangle + \langle \mathbf{u_3},\ 2i\mathbf{u_2} - (1+i)\mathbf{u_3}\rangle$$

$$= (2-i)(0+0) + (3+i)(2i-0) - (1+i) = 6i - 2 - 1 - i = -3 + 5i$$