Matlab in Math 461, part six

QR decomposition

By default, matlab computes a fancier QR decomposition than that given in Lay. If A is an $m \times n$ matrix, the command `[P S] = qr(A)` will return an $m \times m$ orthogonal matrix P and an $m \times n$ upper triangular matrix S so that $A = PS$. If A has rank n, then the first n columns of P will be an orthonormal basis for the column space of A and the last $m - n$ columns will be an orthonormal basis for its orthogonal complement, which is the null space of A^T. The last $m - n$ rows of S will be zero.

In contrast, for the usual QR decomposition given in Lay, A must have rank n, Q is the first n columns of P, and R is the first n rows of S.

So the qr matlab command is more general since it applies to any matrix and gives more information. But it is not the same as the QR decomposition given in Lay, which must be extracted from its results as indicated above. Matlab gives another form of the qr command which will do this extraction for you. If A has rank n, then the command `[Q R] = qr(A,0)` will give the QR decomposition of A, in the sense of Lay.

Other commands

If A is a matrix then `orth(A)` gives a matrix whose columns form an orthonormal basis for the column space of A. Also `Null(A)` gives a matrix whose columns form an orthonormal basis for the Null space of A.

Determining whether or not a subspace is contained in another

Suppose you have two subspaces S and T of $\mathbb{R}^n$. How can you tell whether they are the same? One way to tell is to test whether or not S is contained in T. If it is, and S and T have the same dimension, then $S = T$.

So how can you tell when one subspace is contained in another? Subspaces are generally specified in two different ways. They may be specified explicitly as the Span of a bunch of vectors, or they may be specified implicitly as the solution of some homogeneous equations. In other words, they could be specified as the column space of a matrix A or as the Null space of a matrix B. Note that the Matlab command `Null(B)` changes an implicit specification to an explicit one, since `Null(B)` is a matrix whose column space is the null space of B.

You can find the dimension of the column space or null space of a matrix by computing its rank or counting the number of columns in `orth(A)` or `Null(A)`.

To check whether the column space of A is contained in the null space of B, just compute BA. If $BA = 0$ then each column of A is in the null space of B, so the column space of A is contained in the null space of B. On the other hand, if $BA \neq 0$ then the column space of A is not contained in the null space of B.

To check whether the column space of A is contained in the column space of B you can check whether the orthogonal projection of each column of A to the column space of B is equal to itself. Using equation (5) on page 399 we see that the matrix of orthogonal projection to the column space of B is given in Matlab by `orth(B)*orth(B)'`, (and in fact this formula works in the complex case also). Thus the column space of A is contained in the column space of B if and only if `orth(B)*orth(B)'*A=A`.

Note that another way to check whether two subspaces S and T are equal is to compute the matrices of orthogonal projection $proj_S$ and $proj_T$ and see if these projection matrices are equal.

Finding the orthogonal complement $W^\perp$ of a subspace W

If the subspace W is given explicitly then you can write $W = \text{Col } A$ for some matrix A. But then by Theorem 3 on page 381 you know that $W^\perp = \text{Nul } A^T$. If the subspace W is given implicitly then you can write $W = \text{Nul } B$ for some matrix B. then again by Theorem 3 on page 381 you know that $W^\perp = \text{Col } B^T$.

Problems due April 26

As usual you may work in groups of two or three but no more.

Problem 1: (15) Consider the following subspaces of $\mathbb{R}^5$:

$$S = \text{Span}\{[1\ 0\ 4\ 2\ 6]^T, [2\ 0\ 7\ 1\ 14]^T, [2\ 0\ 6\ \ -2\ 16]^T, [1\ 0\ 3\ \ -1\ 8]^T\}$$

$$T = \text{Span}\{[1\ 0\ 4\ 2\ 6]^T, [1\ 1\ 1\ 1\ 1]^T\}$$

$$W = \{[x_1\ x_2\ x_3\ x_4\ x_5]^T : x_1 + 4x_3 + 2x_4 + 6x_5 = 0,\ x_3 + 3x_4 - 2x_5 = 0\}$$

a) What are the dimensions of S, T, W, $S^\perp$, $T^\perp$, and $W^\perp$?
b) Are any of the six subspaces in a) contained in any others? If so, which ones?
c) Are any of the six subspaces in a) equal to any others? If so, which ones?
d) Find $proj_S + proj_{S^\perp}$, $proj_T + proj_{T^\perp}$, and $proj_W + proj_{W^\perp}$. Come up with a conjecture based on your results.
e) Find the traces of $proj_S$, $proj_T$, and $proj_W$. Come up with a conjecture based on your results.

Problem 2: (5) Let A be a random 4×4 complex matrix with rank 2 and nonzero imaginary part. Check that its rank is in fact 2 before proceeding. Find a QR decomposition of A and check that $A = QR$, and that Q is unitary (recall a matrix U is unitary if $U^{-1} = U^*$). Note, I am asking for matlab's QR decomposition which differs from Lay's QR factorization. (Moreover, Lay asks that the columns of A be linearly independent to have a QR factorization.) Which columns of Q will form a basis for the column space of A?