MATH 461 EXAM #1 Problem 1 March 2, 2005

Name: TA: Section:

1. (39) Suppose a matrix A has an echelon form $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and B has an echelon form $\begin{bmatrix} 2 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Answer the following questions. If there is not enough infromation to give an answer, say so.

a) How many pivots does A have?

Answer: 3

b) How many pivots does B have?

Answer: 4

c) How many solutions does $Ax = 0$ have?

Answer: It has infinitely many solutions since there is a free variable

d) How many solutions does $Bx = 0$ have?

Answer: one, since there is no free variable

e) How many solutions does $Ax = [1\ 2\ 3\ 0]^T$ have?

Answer: You can't tell. There are either infinitely many or no solutions depending on whether $[1\ 2\ 3\ 0]^T$ is in the span of the columns of A or not. But since we don't know A we can't tell.

f) How many solutions does $Bx = [1\ 2\ 3\ 0]^T$ have?

Answer: one, since the columns of B span $\mathbb{R}^4$ and there are no free variables.

g) Is the linear transformation $x \mapsto Ax$ one to one?

Answer: no, since there is a free variable

h) Is the linear transformation $x \mapsto Bx$ one to one?

Answer: yes

i) Is the linear transformation $x \mapsto Ax$ onto?

Answer: no, it does not map onto $\mathbb{R}^4$ because there is not a pivot in every row.

j) *Is the linear transformation $x \mapsto Bx$ onto?*

Answer: yes

k) *Which of A or B is invertible?*

Answer: B

l) *One solution of $Ax = [1\ 2\ 3\ 4]^T$ is $x = [0\ 1\ 0\ 1]^T$. Find all solutions.*

Answer: We can find the solutions of $Ax = 0$ from its echelon form, it is the span of $[-3\ \ -2\ 1\ 0]^T$. So all solutions of $Ax = [1\ 2\ 3\ 4]^T$ are $x = [0\ 1\ 0\ 1]^T + x_3[-3\ \ -2\ 1\ 0]^T$.

m) *One solution of $Bx = [1\ 2\ 3\ 4]^T$ is $x = [0\ 1\ 2\ \ -1]^T$. Find all solutions.*

Answer: There is only one solution, (since $x \mapsto Bx$ is one to one) so $x = [0\ 1\ 2\ \ -1]^T$ is the only solution.

2. *(36)*

a) *Suppose A, B, and C are invertible 4×4 matrices. Solve for the 4×4 matrices W, X, Y and Z.*

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \begin{bmatrix} 0 & A \\ B & C \end{bmatrix} = I_8$$

Answer: We can write $I_8 = \begin{bmatrix} I_4 & 0 \\ 0 & I_4 \end{bmatrix}$ so multiplying out the left hand side we get:

$$\begin{bmatrix} YB & XA + YC \\ WB & ZA + WC \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Since $WB = 0$ we get $W = 0B^{-1} = 0$. Since $YB = I$ we get $Y = B^{-1}$. Also $ZA + WC = I$ so $ZA = I - WC = I$, so $Z = A^{-1}$. Finally, $XA + YC = 0$ so $XA = -YC = -B^{-1}C$, which means $X = -B^{-1}CA^{-1}$. (We did not need to require C to be invertible to get this.)

b) *Find $\begin{bmatrix} 0 & A \\ B & C \end{bmatrix}^{-1}$, with A, B, and C invertible.*

Answer: The calculation above works for any sized A, B, and C so

$$\begin{bmatrix} 0 & A \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{bmatrix}$$

c) *Suppose that* $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ *is a linear transformation and also* $T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\2\\3\end{bmatrix}$ *and* $T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}1\\0\\1\end{bmatrix}$ *and* $T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}2\\2\\4\end{bmatrix}$. *Find* $T(\begin{bmatrix}2\\-1\end{bmatrix})$. *Also find the standard matrix of* T.

Answer:

$$T(\begin{bmatrix}2\\-1\end{bmatrix}) = T(2e_1 - e_2) = 2T(e_1) - T(e_2) = 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\4\\5\end{bmatrix}$$

The standard matrix is $\begin{bmatrix}1 & 1\\2 & 0\\3 & 1\end{bmatrix}$.

3. *(25) Let* $\mathbf{v_1} = [1\ 2\ 3\ 1]^T$, $\mathbf{v_2} = [1\ 0\ -1\ 1]^T$, *and* $\mathbf{v_3} = [1\ 8\ h\ 1]^T$.

a) *Find all* h *so that* $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ *is linearly dependent.*

Answer: We know that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ *is linearly dependent if and only if the matrix* $[v_1\ v_2\ v_3]$ *has less than three pivots. So we can row reduce* $[v_1\ v_2\ v_3]$.

$$\begin{bmatrix}1 & 1 & 1\\2 & 0 & 8\\3 & -1 & h\\1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & -2 & 6\\0 & -4 & h-3\\0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & -2 & 6\\0 & 0 & h-15\\0 & 0 & 0\end{bmatrix}$$

So the only h *where* $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ *is linearly dependent is* $h = 15$.

b) *For each* h *you found in part a), determine if possible weights* c_1 *and* c_2 *so that* $\mathbf{v_3} = c_1\mathbf{v_1} + c_2\mathbf{v_2}$.

Answer: Setting $h = 15$ *and reducing further we get:*

$$\begin{bmatrix}1 & 1 & 1\\0 & -2 & 6\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & 1 & -3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 4\\0 & 1 & -3\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

So a solution to $[v_1\ v_2\ v_3]x = 0$ *is* $x = [-4\ 3\ 1]^T$ *which means that* $-4v_1 + 3v_2 + v_3 = 0$. *Consequently,* $v_3 = 4v_1 - 3v_2$ *so we must have* $c_1 = 4$ *and* $c_2 = -3$. *You could also solve this by solving* $[v_1\ v_2]c = v_3$ *which has the augmented matrix* $[v_1\ v_2\ v_3]$ *so you can read directly from the above reduced echelon form that* $c_1 = 4$ *and* $c_2 = -3$.

c) *For each* h *you found in part a), determine whether or not* $\mathbf{v_3}$ *is in* $\mathrm{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$.

Answer: Since we saw in part b) that for $h = 15$, v_3 *is a linear combination of* v_1 *and* v_2 *we know that* v_3 *is in* $\mathrm{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$ *when* $h = 15$.