MATH 461 EXAM # 2 Problem 1 answers March 30, 2005

Name: TA: Section:

For each of the following sets H, determine whether or not H is a subspace (and give adequate reasons for your answer). If it is a subspace, find a basis if possible and determine the dimension of H.

1a) [8] H is the set of $[x\ y\ z]^T$ in $\mathbb{R}^3$ so that $x + 2y + 4z^2 = 0$.

Answer: *This is not a subspace. For example,* $[0\ {-2}\ 1]^T$ *is in* H *(since* $0{+}2(-2){+}4(1)^2 = 0$*) but* $2 \cdot [0\ \ -2\ 1]^T = [0\ \ -4\ 2]^T$ *is not in* H *(since* $0 + 2(-4) + 4(2)^2 = 8 \neq 0$*).*

1b) [8] H is the set of diagonal 2×2 matrices in $\mathbb{M}_{2\times 2}$.

Answer: *This is a subspace because the sum of two diagonal matrices is diagonal (*$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ 0 & b+d \end{bmatrix}$*) and the product of a scalar and a diagonal matrix is diagonal (*$c\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bc \end{bmatrix}$*) and the zero matrix is diagonal. Since any diagonal matrix* $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ *is a linear combination* $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ *is linearly independent we know a basis of the diagonal* 2×2 *matrices is* $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$*. Since there are two basis vectors, we know* H *is 2 dimensional.*

1c) [8] H is the set of polynomials $p(t)$ in $\mathbb{P}$ so that $\int_1^2 p(t)\,dt = 0$.

Answer: *H is a subspace. If p and q are in H then $p+q$ is in H because* $\int_1^2 (p+q)(t)\,dt = \int_1^2 p(t)\,dt + \int_1^2 q(t)\,dt = 0 + 0 = 0$*. Also, for any scalar c we know cp is in H because* $\int_1^2 (cp)(t)\,dt = c\int_1^2 p(t)\,dt = c \cdot 0 = 0$*. Finally, the zero function is in H because* $\int_1^2 0\,dt = 0$*. But H has no basis because it is infinite dimensional. To see this, note that for each positive integer n we may find a c_n so $t^n - c_n$ is in H (*$c_n = (2^{n+1} - 1)/(n+1)$*) and the sets* $\{t - c_1, t^2 - c_2, \ldots, t^n - c_n\}$ *are linearly independent. So there are arbitrarily large linearly independent sets in H which means H could not be finite dimensional.*

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Let $T: \mathbb{P}_3 \to \mathbb{P}_2$ be the transformation $T(p) = p' - p(0)$, so for example $T(t^3 - t + 2) = 3t^2 - 1 - 2 = 3t^2 - 3$.

2a) [8] Show that T is a linear transformation.

Answer: *Note that* $T(p+q) = (p+q)' - (p+q)(0) = p' + q' - p(0) - q(0) = T(p) + T(q)$. *Also* $T(cp) = (cp)' - (cp)(0) = cp' - cp(0) = cT(p)$. *So* T *is a linear transformation.*

2b) [6] Find a basis for the kernel of T.

Answer: *The kernel of* T *is all* p *in* $\mathbb{P}_3$ *so that* $T(p) = 0$. *Any* p *in* $\mathbb{P}_3$ *can be written as* $p(t) = at^3 + bt^2 + ct + d$ *for some scalars* a, b, c, d. *Then* $T(p) = 3at^2 + 2bt + c - d$. *Setting* $T(p) = 0$ *we get* $3a = 0$, $2b = 0$, *and* $c - d = 0$. *So* $a = b = 0$ *and* $c = d$. *So* $p(t) = dt + d = d(t+1)$. *So we see that the kernel of* T *is the span of* $\{t+1\}$ *so a basis of this kernel is* $\{t+1\}$.

2c) [6] Find the dimension of the kernel of T and dimension of the range of T.

Answer: *From the basis of part b) we see the dimension of the kernel of* T *is 1. For the dimension of the range you can either us the fact stated in class that if* $T: V \to W$ *then* $\dim(kernel T) + \dim(range T) = \dim V$ *to see that* $\dim(range T) = 4 - 1 = 3$, *or you can note that* T *is onto since* $T(at^3/3 + bt^2/2 + ct) = at^2 + bt + c$ *and thus the range of* T *is all of* $\mathbb{P}_2$ *which has dimension 3.*

2d) [6] Is T one to one?________ Is T onto?________ Give reasons for your answers below.

Answer: T *is not one to one since its kernel is nonzero.* T *is onto, which you can either see directly as we did in part c or else use the calculation that the range has dimension 3 to conclude that the range must then be all of the three dimensional space* $\mathbb{P}_2$.

MATH 461 EXAM # 2 Problem 3 answers March 30, 2005

Name: TA: Section:

Let $A = \begin{bmatrix} 1 & 2 & 3 & 9 \\ 2 & 0 & -2 & 6 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 7 \end{bmatrix}$. You may use the following matlab output in this question:

EDU>A = [1 2 3 9; 2 0 -2 6; 0 1 2 3; 1 2 3 7];

EDU>rref(A)

ans =

$$\begin{matrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

3a) [6] What is the rank of A?_______ What is the rank of A^T?_______

Answer: *Since there are three pivots, the rank of A is 3. Since A and A^T always have the same rank, the rank of A^T is also 3.*

3b) [6] Find a basis of the column space of A.

Answer: *The pivot columns are 1, 2, and 4 so these columns of A form a basis of the column space,* $\{[1\ 2\ 0\ 1]^T, [2\ 0\ 1\ 2]^T, [9\ 6\ 3\ 7]^T\}$.

3c) [6] Find a basis of the row space of A.

Answer: *The pivot rows of the echelon form make a basis* $\{[1\ 0\ -1\ 0]^T, [0\ 1\ 2\ 0]^T, [0\ 0\ 0\ 1]^T\}$.

3d) [8] Find a basis of the null space of A.

Answer: *From the echelon form we see a basis for the null space is* $\{[1\ \ -2\ 1\ 0]^T\}$.

3e) [4] What is the determinent of A?_______

Answer: *Since the rank is less than 4 we know A is not invertible so its determinent must be 0.*

MATH 461 EXAM # 2 Problem 4 answers March 30, 2005

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4a) [6] Is $\left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix} \right\}$ a linearly independent set in the vector space $\mathbb{M}_{2\times 2}$ of 2×2 matrices? You must give an adequate reason for your answer.

Answer: *Yes, this set is linearly independent because the two matrices are not multiples of each other. (Remember a set of two vectors is linearly independent if and only if the two vectors are nonzero and not multiples of each other.) The fact that the columns of each matrix are linearly dependent is irrelevant. In this vector space, the vectors are matrices.*

4b) [6] Is $\{1, \cos^2(t), \sin^2(t)\}$ a linearly independent set of functions defined on $\mathbb{R}$? You must give an adequate reason for your answer.

Answer: *No, this set is linearly dependent because, for example,* $1 = 1 \cdot \cos^2(t) + 1 \cdot \sin^2(t)$ *so the first vector is a linear combination of the others.*

4c) [8] Find a basis $\mathcal{B}$ for the span of $\{t^2,\ t-1,\ t^2+2t-2\}$ in $\mathbb{P}_2$. Find the coordinates of t^2+5t-5 relative to your basis $\mathcal{B}$.

Answer: *These vectors are linearly dependent because* $t^2+2t-2 = t^2+2(t-1)$. *So we may throw out* t^2+2t-2 *and the span remains the same. So* $Span\{t^2,\ t-1,\ t^2+2t-2\} = Span\{t^2,\ t-1\}$. *But* $\{t^2,\ t-1\}$ *is a linearly independent set so it is a basis for its span. So we may choose* $\mathcal{B} = \{t^2,\ t-1\}$. *(Actually in this problem, any two of the three vectors would be a basis.) Since* $t^2+5t-5 = 1 \cdot t^2 + 5(t-1)$ *we know that* $[t^2+5t-5]_{\mathcal{B}} = [1\ 5]^T$.