MATH 461 EXAM # 3 Problem 1 April 29, 2005

Name: TA: Section:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

1a) [20] Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$.

Answer: *The normal equation is $A^T Ax = A^T b$. Recall that $A^T A$ is the square matrix of dot products of the columns of A with each other and $A^T b$ is the dot products of the columns of A with b, so the normal equation is:*

$$\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Solving this by gaussian elimination gives:

$$\begin{bmatrix} 2 & 2 & 6 \\ 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 6 \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{bmatrix}$$

So $x = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ is the only least squares solution.

1b) [5] Find the orthogonal projection of $\mathbf{b}$ to the column space of A.

Answer: *If x is a least squares solution then the projection $\hat{b}$ of b to the column space of A is Ax. So in this case the projection is*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Note that since the columns of A are not orthogonal, the projection is **not** *given by the formula on page 395 with u_i the columns of A.*

MATH 461 EXAM # 3 Problem 2 April 29, 2005

Name: TA: Section:

Consider the vector space $C[0,1]$ of continuous real valued functions on the interval $[0,1]$ with inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Let H be the subspace spanned by the functions 1 and t. Let $T\colon H \to H$ be differentiation, $T(f) = df/dt$.

2a) [10] Find an orthonormal basis $\mathcal{B}$ for H.

Answer: *Using the Gram Schmidt process to get an orthogonal basis first, we set* $u_1 = 1$ *and then set* $u_2 = t-(\langle t,1\rangle/\langle 1,1\rangle)1$. *We have* $\langle t,1\rangle = \int_0^1 t\,dt = 1/2$ *and* $\langle 1,1\rangle = \int_0^1 1\,dt = 1$, *so* $u_2 = t-1/2$. *So an orthogonal basis of* H *is* $\{1, t-1/2\}$ *to make it orthonormal, calculate* $\langle 1,1\rangle = \int_0^1 1\,dt = 1$ *and* $\langle t-1/2, t-1/2\rangle = \int_0^1 (t-1/2)^2\,dt = (t-1/2)^3/3]_0^1 = (1/2)^3/3 - (-1/2)^3/3 = 1/12$. *So an orthonormal basis is* $\{1/\sqrt{1}, (t-1/2)/\sqrt{1/12}\} = \{1, 2\sqrt{3}t-\sqrt{3}\}$. *Let us call* $v_1 = 1$ *and* $v_2 = 2\sqrt{3}t - \sqrt{3}$.

2b) [10] Find the matrix of T relative to the basis $\mathcal{B}$ you found in part a).

Answer: *We have* $T(v_1) = T(1) = 0 = 0v_1 1 + 0v_2$ *and* $T(v_2) = T(2\sqrt{3}t - \sqrt{3}) = 2\sqrt{3} = 2\sqrt{3}v_1 + 0v_2$. *So the matrix of* T *relative to the basis* $\mathcal{B}$ *is* $\begin{bmatrix} 0 & 2\sqrt{3} \\ 0 & 0 \end{bmatrix}$.

2c) [5] Find the orthogonal projection of $f(t) = t^3$ to H.

Answer: *This is* $\langle t^3, v_1\rangle v_1 + \langle t^3, v_2\rangle v_2$. *We have*

$$\langle t^3, v_1\rangle = \int_0^1 t^3\,dt = 1/4$$

$$\langle t^3, v_2\rangle = \int_0^1 t^3(2\sqrt{3}t - \sqrt{3})\,dt = 2\sqrt{3}t^5/5 - \sqrt{3}t^4/4]_0^1 = 2\sqrt{3}/5 - \sqrt{3}/4 = 3\sqrt{3}/20$$

So the projection is

$$1/4 + 3\sqrt{3}/20(2\sqrt{3}t - \sqrt{3}) = 9t/10 - 1/5$$

An alternative calculation uses the orthogonal basis to avoid square roots: $\langle t^3, u_1\rangle = 1/4$ *and* $\langle t^3, u_2\rangle = \int_0^1 t^3(t-1/2)\,dt = 3/40$. *Then the projection by the formula on page 395 is:*

$$((1/4)/1)1 + ((3/40)/(1/12))(t-1/2) = 1/4 + (9/10)(t-1/2) = 9t/10 - 1/5$$

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Name: TA: Section:

Suppose there are square matrices A_1, A_2, and A_3. The characteristic polynomial of A_1 is $(t^2-1)(t^2+2t+5)$. The characteristic polynomials of A_2 and A_3 are the same, $(t-1)(t+6)t^2$. All of the eigenspaces of A_2 have dimension 1. One of the eigenspaces of A_3 has dimension bigger than 1.

3a) [8] Find all the eigenvalues of each A_j.

Answer: *The eigenvalues are the roots of the characteristic polynomial. So the eigenvalues of* A_1 *are the roots of* t^2-1 *(which are* ± 1*) and the roots of* t^2+2t+5 *which are* $(-2\pm\sqrt{2^2-4\cdot 5})/2=-1\pm 2\mathbf{i}$. *So the eigenvalues of* A_1 *are* $1,-1,-1+2\mathbf{i},-1-2\mathbf{i}$. *The eigenvalues of* A_2 *and* A_3 *are the same,* $1,-6,0$ *where* 0 *has multiplicity 2.*

3b) [6] Which of the A_i are diagonalizable, that is, for which A_i is there a real matrix P so that $P^{-1}A_iP$ is diagonal?

Answer: *While* A_1 *is diagonalizable using a complex* P *it is not diagonalizable using a real* P *since it has complex eigenvalues and hence complex eigenvectors. Since the columns of* P *need to be eigenvectors of* A, P *could not be real. We know* A_2 *is not diagonalizable by Theorem 7b on page 324, since the dimension of the 0 eigenspace is not 2. But by c) below, all eigenspaces of* A_3 *have dimension equaling the multiplicity, so* A_3 *is diagonalizable.*

3c) [3] Which eigenspace of A_3 has dimension bigger than 1 and what is its dimension?

Answer: *By Theorem 7 on page 324, an eigenvalue with an eigenspace of dimension bigger than one must have multiplicity bigger than 1. So the eigenvalue must be 0. Again by thm 7 the maximum dimension is the multiplicity, 2. So the dimension of the 0 eigenspace is exactly 2.*

3d) [3] Which of the A_i are invertible?

Answer: *Since the null space of a matrix is its 0 eigenspace, a square matrix is invertible if and only if 0 is not an eigenvalue. So* A_1 *is invertible and* A_2 *and* A_3 *are not.*

MATH 461 EXAM # 3 Problem 4 April 29, 2005

Name: TA: Section:

For each of the following matrices:

+ Find all of its eigenvalues and an eigenvector for each eigenvalue.
+ If possible, find a (possibly complex) matrix P and a diagonal matrix D so that the given matrix equals PDP^{-1}.
+ If possible, find a real matrix Q so that the given matrix is QCQ^{-1} where C is of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

4a) [15] $\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}$

Answer: *The characteristic polynomial is* $t^2 - 4t + 4 = (t-2)^2$ *so the only eigenvalue is 2. The 2 eigenspace is the null space of* $\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix} - 2I_2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ *Using Gaussian elimination:*

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the null space is the span of $(1\ 2)^T$. *So* $(1\ 2)^T$ *is an eigenvector for the eigenvalue 2. But since* $\mathbb{R}^2$ *does not have a basis of eigenvectors we know* P *and* D *cannot be found, (since the columns of* P *must be a basis of eigenvectors). Likewise,* Q *and* C *only exist for non real eigenvalues.*

4b) [15] $\begin{bmatrix} 4 & -1 \\ 5 & 0 \end{bmatrix}$

Answer: *The characteristic polynomial is* $t^2 - 4t + 5$ *which has roots*

$$t = (4 \pm \sqrt{4^2 - 4 \cdot 5})/2 = 2 \pm \mathbf{i}$$

Consider the eigenvalue $2 - \mathbf{i}$. *Then its eigenspace is the null space of*

$$\begin{bmatrix} 4 & -1 \\ 5 & 0 \end{bmatrix} - (2 - \mathbf{i})I_2 = \begin{bmatrix} 2+\mathbf{i} & -1 \\ 5 & -2+\mathbf{i} \end{bmatrix}$$

Without even doing complex arithmetic we know the rows of this matrix must be linearly dependent, since it is not invertible, so the second row is a multiple of the first. Thus Gaussian elimination gives us:

$$\begin{bmatrix} 2+\mathbf{i} & -1 & 0 \\ 5 & -2+\mathbf{i} & 0 \end{bmatrix} \sim \begin{bmatrix} 2+\mathbf{i} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we see that the null space is the span of $\begin{bmatrix} 1 \\ 2+\mathbf{i} \end{bmatrix}$. *So an eigenvector for* $2 - \mathbf{i}$ *is* $\begin{bmatrix} 1 \\ 2+\mathbf{i} \end{bmatrix}$. *Taking complex conjugates, an eigenvector for* $2+\mathbf{i}$ *is* $\begin{bmatrix} 1 \\ 2-\mathbf{i} \end{bmatrix}$. *So we may take* $P = \begin{bmatrix} 1 & 1 \\ 2+\mathbf{i} & 2-\mathbf{i} \end{bmatrix}$ *and* $D = \begin{bmatrix} 2-\mathbf{i} & 0 \\ 0 & 2+\mathbf{i} \end{bmatrix}$. *To make* Q, *take the real and imaginary parts of an eigenvector, so we may take* $Q = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. *The entries of* C *will be the real and imaginary parts of the eigenvalue, so* $C = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ *or possibly* $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, *I can never remember which. But you did not have to tell me what* C *was. Checking, we see that*

$$Q \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} Q^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & 0 \end{bmatrix}$$

so $C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.