Factoring intervals from bundles

In this exposition we say a topological space X is nice if it is locally compact, normal, and second countable. The theorems are true for paracompact (Hausdorff) spaces too, see Hirsch.

If ξ is a vector bundle and $f: B \to B(\xi)$ is continuous then there is an induced vector bundle $f^*(\xi)$ with $B(f^*(\xi)) = B$, $E(f^*(\xi)) = \{(x, y) \in B \times E(\xi) \mid f(x) = \pi(\xi)(y)\}$, and $\pi(f^*(\xi))(x, y) = x$. A special case is where f is inclusion $B \subset B(\xi)$ then we may take $E(f^*(\xi)) = \pi(\xi)^{-1}(B)$ and $\pi(f^*(\xi))$ the restriction of π. In this case we also denote $\xi|_B = f^*(\xi)$.

If ξ is a vector bundle and X is any space then there is a vector bundle $\xi \times X$ with $E(\xi \times X) = E(\xi) \times X$, $B(\xi \times X) = B(\xi) \times X$, and $\pi(\xi \times X) = \pi(\xi) \times id_X$. Of course $\xi \times X|_{B(\xi)\times x}$ is isomorphic to ξ for each $x \in X$.

Let $I = [0, 1]$.

Theorem. *If B is nice and ξ is a vector bundle over $B \times I$, then all the restricted bundles $\xi|_{B\times t}$ are isomorphic and in fact there is a bundle isomorphism $\varphi: \xi \to (\xi|_{B\times 0}) \times I$.*

Corollary. *If B is nice then homotopic maps $f_0, f_1: B \to B(\xi)$ induce isomorphic bundles $f_i^*(\xi)$.*

Proof: If $F: B \times I \to B(\xi)$ is a homotopy then the theorem says $F^*(\xi)|_{B\times 0}$ is isomorphic to $F^*(\xi)|_{B\times 1}$. But each $F^*(\xi)|_{B\times i}$ is isomorphic to $f_i^*(\xi)$. ■

Corollary. *Any vector bundle over a nice contractible space is trivial.*

Proof: Let $F: B(\xi) \times I \to B(\xi)$ be a homotopy from the identity to a constant map $f(x) = c$. By the previous corollary, the indentity and the constant map induce isomorphic bundles. But $id^*(\xi) = \xi$ and $f^*(\xi)$ is trivial, $E(f^*(\xi)) = B(\xi) \times \pi(\xi)^{-1}(c)$. ■

Before proving the theorem we'll present some technical lemmas.

Lemma 1. *Let ξ be a vector bundle with $B(\xi) = B \times I$. Then each $x \in B$ has a neighborhood V so that $\xi|_{V\times I}$ is trivial.*

Proof: We may pick bundle charts $\varphi_\alpha: V_\alpha \times I_\alpha \times \mathbb{R}^k \to E(\xi)$ so that $x \times I \subset \bigcup V_\alpha \times I_\alpha$ where the I_α are intervals and V_α are open. By compactness of I we only need a finite number of these. Let $V = \bigcap V_\alpha$ (a finite intersection so V is open). Reindex the α to $1, 2, \ldots, n$ and choose $t_i \in I$ so $0 = t_0 < t_1 < \cdots < t_n = 1$ and $[t_{i-1}, t_i] \subset I_i$. Define $\varphi: V \times I \times \mathbb{R}^k \to E(\xi)$ by $\varphi(x, t, y) = \varphi_i(x, t, y')$ if $t_{i-1} \le t \le t_i$ where y' is defined by $\varphi(x, t_{i-1}, y) = \varphi_i(x, t_{i-1}, y')$. ■

Lemma 2. *Let ξ be a vector bundle with $B(\xi) = B \times I$, let $A \subset U \subset B$ with A closed and U open, let $K \subset B$ be compact, and suppose we have a vector bundle isomorphism $\varphi: \xi|_{U\times 0} \times I \to \xi|_{U\times I}$ with $\varphi(z, 0) = z$. Suppose also that B is normal. Then there is a neighborhood V of $A \cup K$ and a vector bundle isomorphism $\psi: \xi|_{V\times 0} \times I \to \xi|_{V\times I}$ so that ψ restricts to φ on $(\xi|_{A\times 0}) \times I$ and $\psi(z, 0) = z$.*

Proof: By Lemma 1 we may cover K by open sets $V_1, V_2, \dots, V_n$ so that there are bundle charts $\mu_i\colon V_i \times I \times \mathbb{R}^k \to \pi(\xi)^{-1}(V_i \times I)$. Pick closed $K_i \subset V_i$ so $K \subset \bigcap_{i=1}^n K_i$. We have maps $\kappa_i\colon (U \cap V_i) \times I \to \mathbf{Gl}(k, \mathbb{R})$ where $\mu_i(x, t, \kappa_i(x,t)y) = \varphi(\mu_i(x, 0, y), t)$. Note $\kappa_i(x, 0) = id$. This κ_i measures the difference between the factorization of the bundle given by φ and that given by μ_i. Since B is normal, we may pick a continuous $\beta\colon B \to [0,1]$ with support in U so that $\beta|_{U''} = 1$ for some neighborhood U'' of A in U. If $U' = U'' \cup V_1$ we have a vector bundle isomorphism $\varphi'\colon \xi|_{U' \times 0} \times I \to \xi|_{U' \times I}$ given by φ on $\xi|_{U'' \times 0} \times I$ and given by $\varphi'(\mu_1(x, 0, y), t) = \mu_i(x, t, \kappa_i(x, t\beta(x))y)$ on $\xi|_{V_1 \times 0} \times I$. Enlarge A to $A \cup K_1$ and do the same with V_2, then V_3 and so on. Eventually we are done. ∎

To prove the theorem note that since B is nice there is a countable collection of compact sets K_i which cover B so that $K_{i+1} \subset \mathrm{Int}K_i$ for all i and K_0 is empty. By Lemma 2 with A empty and $K = K_{2n+1} - \mathrm{Int}K_{2n}$ we have vector bundle isomorphisms $\varphi_{2n+1}\colon (\xi|_{U_{2n+1} \times 0}) \times I \to \xi|_{U_{2n+1} \times I}$ for some neighborhoods U_{2n+1} of $K_{2n+1} - \mathrm{Int}K_{2n}$. Now apply Lemma 2 with $A = \bigcup_i K_{2i+1} - \mathrm{Int}K_{2i}$ and $K = K_{2n} - \mathrm{Int}K_{2n-1}$ to get vector bundle isomorphisms $\varphi_{2n}\colon (\xi|_{U_{2n} \times 0}) \times I \to \xi|_{U_{2n} \times I}$ for some neighborhoods U_{2n} of $K_{2n} - \mathrm{Int}K_{2n-1}$ so $\varphi_{2n}(z, t) = \varphi_i(z, t)$ if $\pi(\xi)(z) \in (K_i - \mathrm{Int}K_{i-1}) \times 0$, $i = 2n \pm 1$. Piece these together to get a vector bundle isomorphism $\varphi\colon (\xi|_{B \times 0}) \times I \to \xi$ by letting $\varphi(z, t) = \varphi_i(z, t)$ if $\pi(\xi)(z) \in (K_i - \mathrm{Int}K_{i-1}) \times 0$. ∎

The basic outline of the above proof is one used often in differential topology. You want some sort of structure on a space (usually a manifold, but here a vector bundle). You prove a relative local version, meaning you assume you have the structure on a neighborhood of a closed set, and then extend this structure to a chart without changing it on the closed set. (We did this in the proof of Lemma 2.) In the compact case we are done, since we can cover with a finite number of charts. But the noncompact case can be proven as above, by writing the space as a union of compact pieces $K_i - \mathrm{Int}K_{i-1}$, putting the structure on a neighborhood of every other band, then filling in on the remaining bands. Hirsch even formalizes this in globalization metatheorems but I find that rather opaque.

Note that there was nothing special about vector spaces used in the proofs above, the same proofs work for fiber bundles, where $\pi\colon E \to B$ is a fiber bundle with any structure group, see pages 106-107 of Bredon for definitions.