Differential topology notes

Following is what we did each day with references to the bibliography at the end.

9/3: We defined smooth manifolds is 3 ways- patching charts [**Br**, 68], functional structure [**Br**, 69], and submanifold of R^N [**BJ**, 9] (I can't seem to find this in [Br]. X is a submanifold of R^N if for each x in X, there is an open neighborhood U and a diffeomorphism $g : U \to V$ where V is open in R^N and $g^{-1}(R^n) = U \cap X$.) You can define C^k manifolds for $k = 0, 1, \dots, \infty, \omega$ also piecewise linear, Lipchitz, affine, complex manifolds and many other kinds but we will stick to C^∞. Smooth will always mean C^∞. Any C^k manifold $k = 1, 2, \dots$ has a unique smooth structure and any smooth manifold has a unique real analytic (C^ω) structure (we'll see these facts later when we know transversality and normal bundles). However some C^0 manifolds of dimension ≥ 4 have no smooth structure or have many different smooth structures (probably beyond the scope of this course). We then gave examples of manifolds in some low dimensions.

- dim 0 Countable unions of points with the discrete topology. (complete classification)
- dim 1 Countable unions of lines ($\mathbb{R}^1$) and circles (S^1). (complete classification, [**Mi1**, 55])
- dim 2 Any compact connected 2 dimensional manifold is diffeomorphic to either a sphere connected sum some tori ($S^1 \times S^1$) or the connected sum of projective planes ($\mathbb{RP}^2$) [**Ki**, 79]. We also gave some noncompact examples, $\mathbb{R}^2$ and the Mobius band and an infinite connected sum of tori. Any open subset of a manifold is a manifold so that gives lots of examples.

9/5: We looked at ways of generating smooth manifolds.

- Any open subset of a smooth manifold is naturally a smooth manifold. (just restrict charts)
- The cartesian product of two manifolds is a manifold. (take products of charts)
- If you glue two manifolds along diffeomorphic open subsets and the result is Hausdorff, then you have a smooth manifold. (cover by charts in one or the other manifold)
- The connected sum of two manifolds is a manifold. Note: we will show later that there are at most two possibilities for the connected sum of two connected manifolds, despite the choices involved in the construction.
- If $f : M \to N$ is a smooth map between smooth manifolds, then for almost all points $y \in N$, the inverse image $f^{-1}(y)$ is a manifold. In fact, for almost all smooth submanifolds $Y \subset N$, $f^{-1}(Y)$ is a submanifold of X. (We will see why when we know more about transversality.) For example, the solutions to a random set of smooth equations in $\mathbb{R}^n$ is a submanifold.
- If M is a manifold and G is a group of self-diffeomorphisms of M, then sometimes M/G is a smooth manifold. For example this is true if each point of M has a neighborhood U so $U \cap g(U)$ is empty for each nonidentity $g \in G$. Examples, $\mathbb{RP}^n = S^n/G$ where G is the 2 group generated by the antipodal map. $L(p, q)$ the lens space S^3/G where G is the cyclic group of order p generated by $(z, w) \mapsto (ze^{2\pi i/p}, we^{2q\pi i/p})$, where p and q are relatively prime and z, w are complex with $|z|^2 + |w|^2 = 1$. Another example is $\mathbb{C}^{n+1} - 0/G = \mathbb{CP}^n$ where G is the diffeomorphisms of the form $(z_0, z_1, \dots, z_n) \mapsto (\omega z_0, \omega z_1, \dots, \omega z_n)$ where $\omega \in \mathbb{C} - 0$.
- Many natural spaces in mathematics end up being manifolds. For example, $O(n)$ the space of orthogonal $n \times n$ matrices is a smooth submanifold of $\mathbb{R}^{n^2}$. The Grassmanian $G(n, k)$ of k planes in n space is a smooth manifold. One way to describe $G(n, k)$ is as the set of $n \times n$ matrices P so $P = P^T$, $P^2 = P$, and trace$P = k$, here P is the matrix of orthogonal projection to a k dimensional linear subspace. Another way is the quotient. $O(n)/O(k) \times O(n - k)$ where we quotient out by matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A \in O(k)$ and $B \in O(n - k)$.

9/8 We looked at the tangent space to smooth manifold. If $\dim M = m$ and $p \in M$, then $T_p(M)$ is an m dimensional vector space which we can think of as an infinitesimal view of the manifold at p. We have the tangent bundle $T(M) = \bigcup_{p \in M} T_p(M)$ with some topology which we explain below. If M is a smooth curve or surface in space we have the notion of tangent line or plane from multivariable calculus. We could define the tangent plane to a surface at p as the union of tangent vectors to curves through p. So for a general smooth manifold M we could define $T_p(M)$ to be equivalence classes for some equivalence relation on the space of curves through p. This is useful for thinking of individual tangent vectors but not for $T_p(M)$ (for example the vector sum is not at all natural.) Instead, use gluing. If $U \subset \mathbb{R}^m$ is open, then $T(U) = U \times \mathbb{R}^m$ where each $\mathbb{R}^m$ factor is a vector space. If $\varphi: U \to \mathbb{R}^m$ and $\psi: V \to \mathbb{R}^m$ are two charts then $T(U) = U \times \mathbb{R}^m$ and $T(V) = V \times \mathbb{R}^m$ and we need to figure out how to glue them together to get $T(U \cup V)$. If $p \in U \cap V$ and $\alpha: (\mathbb{R}, 0) \to (\mathbb{R}^m, \varphi(p))$ is a smooth curve with tangent vector $\alpha'(0)$ at $\varphi(p)$, then by the chain rule, $D(\psi\varphi^{-1})\alpha'(0) = (\psi\varphi^{-1}\alpha)'(0)$ where Dh stands for the Jacobian matrix of partial derivatives of a map h. Thus to get $T(U \cup V)$ we identify $(p, x) \in U \times \mathbb{R}^m$ with $(p, D(\psi\varphi^{-1})x) \in V \times \mathbb{R}^m$. Gluing all the charts of M in this way gives us $T(M)$. A third view of a tangent vector is as a directional derivative. If $v \in T_p(M)$, represent v by some curve $\alpha: (\mathbb{R}, 0) \to (M, p)$. Then for any $f \in C^\infty(M, \mathbb{R})$ we let $v(f) = d(f \circ \alpha(t))/dt|_{t=0}$. Note v is linear and satisfies the product rule $v(fg) = f(p)v(g) + g(p)v(f)$. You could define $T_p(M)$ as all linear maps satisfying the product rule which are local (so $v(f) = v(g)$ if $f(x) = g(x)$ for all x near p). See [**BJ**, 13-21], [**Br**, 76-79], [**H**, 17]. We often abbreviate $T(M) = TM$ and $T_p(M) = T_pM$. If $h: M \to N$ is smooth we have a map $dh: TM \to TN$. In local coordinates we may suppose $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ and then $dh(x, y) = (h(x), Dh(y))$ where Dh as before is the $n \times m$ matrix of first partials. We have $d(h \circ g) = dh \circ dg$. also if $f: M \to \mathbb{R}$, then $df(v) = v(f)$. The map dh gives the local linear part of h since (in $\mathbb{R}^m$), $h(x) \approx h(p) + Dh \cdot (x - p)$. Examples $T(S^1)$ is trivial, $T(S^1) = S^1 \times \mathbb{R}$. The tangent bundle to the Möbius band is nontrivial. The tangent bundle of S^2 is nontrivial for a subtle reason.

9/10 We define vector bundles, $\pi: E \to B$ is a rank k vector bundle if for each $x \in B$, $\pi^{-1}(x)$ is a vector space of dimension k and there is a neighborhood U of x and a homeomorphism $\psi: U \times \mathbb{R}^k \to \pi^{-1}(U)$ so that $\pi\psi(y, z) = y$ and for each $y \in U$, the map $z \mapsto \psi(y, z)$ is an invertible linear transformation from $\mathbb{R}^k$ to $\pi^{-1}(y)$. We call this ψ a bundle chart. Thus TM is a vector bundle. Two vector bundles $\pi: E \to B$ and $\pi': E' \to B'$ are isomorphic if there are homeomorphisms $h: E' \to E$ and $g : B' \to B$ so $\pi h = g\pi'$ and each $h: \pi'^{-1}(x) \to \pi^{-1}(g(x))$ is a vector space isomorphism. (Or sometimes if $B = B'$ we require that g be the identity.) If $f: B' \to B$ is continuous there is an induced bundle with total space $E' = \{(x, y) \in B' \times E \mid f(x) = \pi(y)\}$ and projection $(x, y) \mapsto x$. If $B' \subset B$ we may restrict the bundle to B' which is isomorphic to the induced bundle of the inclusion map. Next time we will see that if the base space of a bundle is $B \times [0, 1]$ for B nice, then the bundle splits as a bundle cross $[0, 1]$. As a consequence, homotopic maps induce isomorphic bundles, and bundles over contractible spaces are trivial. If $\psi_i: U_i \times \mathbb{R}^k \to \pi^{-1}(U_i)$, $i = 0, 1$ are two bundle charts then we get a gluing map $\Psi: U_0 \cap U_1 \to \mathrm{Gl}(k, \mathbb{R})$ where $\Psi(x)$ is the linear isomorphism ($k \times k$ nonsingular matrix) so that $\psi_1^{-1}\psi_0(x, y) = (x, \Psi(x)y)$. By restricting these gluing maps to various subgroups of $\mathrm{Gl}(k, \mathbb{R})$ we get different types of bundles. For example orientable vector bundles come from the subgroup of positive determinent matrices. Complex vector bundles come from $\mathrm{Gl}(k/2, \mathbb{C})$. It turns out $O(k)$ bundles give us nothing new since any vector bundle over a paracompact base space can be given an $O(k)$ structure (essentially because $\mathrm{Gl}(k, \mathbb{R})$ deformation retracts to $O(k)$). Stated without hint of proof is the existence of a universal bundle $EO(k) \to BO(k)$ so that every vector bundle over a nice space is a pullback of this bundle via a map to $BO(k)$ and in fact there is a one to one correspondence between isotopy classes of rank k vector bundles over B and homotopy classes of maps from B to $BO(k)$. References [**Br**, 106-114], [**BJ**, 22-31], [**H**, 85-91]

9/12 We show that any vector bundle over $B \times I$ with B nice splits as a bundle cross I, see [**H**, 90] or the writeup I handed out in class.

9/15 We classify rank k vector bundles over S^n as being in one to one correspondence with homotopy classes of maps from (S^{n-1}, x_0) to $(O(k), id)$, which is $\pi_{n-1}(O(k), id)$ if you know about higher homotopy groups. For example, for each k there are only two isomorphism classes of bundles over S^1 (since $O(k)$ has just two path components). For each $k > 2$ there are two rank k bundles over S^2, there is a $\mathbb{Z}$'s worth of isomorphism classes of rank 2 bundles over S^2, and there is just one line bundle over S^2. (This follows from calculations of $\pi_1(SO(k))$, see [**Br**, 467].)

9/17 We introduced the notion of a section of a bundle, a continuous $s: B \to E$ so that $\pi \circ s = id_B$. For example, a section of the tangent bundle of a smooth manifold is a vector field. Note, if the base of a vector bundle is a smooth manifold, we can define smooth vector bundles (where all bundle chart gluing maps $U_1 \cap U_2 \to \mathrm{Gl}(k, \mathbb{R})$ are smooth). So we can talk about smooth sections and thus smooth vector fields. Any vector bundle over a smooth manifold has a unique smoothing, as we will see later. Any natural operation on vector spaces gives rise to a corresponding operation on vector bundles. For example if $E \to B$ and $F \to B$ are vector bundles Whitney sum $E \oplus F$ comes from the direct sum of vector spaces, then there is the dual $E^* \to B$, the tensor product $E \otimes F$, $Hom(E, F)$, alternating multilinear forms $Alt^k(E)$ ([**Br**, 260]), quadratic forms (a subbundle of the bilinear forms $(E \otimes E)^*$), and the quotient E/F (If F is a subbundle of E). For example we have the dual tangent space TM^* to a manifold, sections of this bundle are called 1-forms, and in general sections of $Alt^k(TM)$ are called k-forms. I expect we will have more to say about k-forms later. If $M \subset N$ is a smooth submanifold we can define the normal bundle of M in N to be the quotient bundle $TN|_M/TM$, but for submanifolds of $\mathbb{R}^k$ you could just as well define it to be the bundle of vectors in TN which are perpendicular to TM. Note that if $M \subset \mathbb{R}^n$ is a smooth submanifold and $E \to M$ is its normal bundle in $\mathbb{R}^n$, then $TM \oplus E = T(\mathbb{R}^n)|_M$ which is trivial. If $Q \to B$ is the bundle of quadratic forms on E, a section $s\colon B \to Q$ which is always positive definite gives an inner product structure on each vector space fiber. If B is paracompact, a partition of unity argument shows such sections exist. For example, smooth manifolds have Reimannian metrics. Some references [**Br**, 108-114], [**H**, 85-98], [**BJ**, 22-43].

9/22 Let M and N be manifolds of dimension m and n respectively. Suppose $f\colon M \to N$ is smooth and the rank of df_x is a constant r at all points $x \in M$. Then locally we can find coordinates so $f(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$ [**BJ**, 45]. Of special interest are when $r = n$, called a submersion, and $r = m$, called an immersion, or $r = m = n$ which is both and also called a local diffeomorphism. If f is a submersion, then each fiber $f^{-1}(x)$ is a smooth submanifold of dimension $m - n$. When we prove this for immersions we see more, namely that the rank $n-m$ bundle $f^*(TN)/TM$ (called the normal bundle of the immersion) has a local diffeomorphism to M which restricts to f on the zero section. If we consider the case of an embedding (an immersion which is also a homeomorphism to its image), we get the existence of a tubular neighborhood of the embedding. Since any submanifold is the image of an embedding (the inclusion is an embedding!) submanifolds have tubular neighborhoods. Later we will show uniqueness of tubular neighborhoods. References are [**Br**, 83,93], [**H**, 21,109], [**BJ**, 44-50,115-124].

9/24 Hirsch in a 1959 TAMS paper showed that (except for the equidimensional case) classifying immersions is just a bundle problem. For example, if $m < n$ or no component of M is compact, then a continuous map $f\colon M \to \mathbb{R}^n$ can be approximated by an immersion if and only if there is a rank $n - m$ bundle $E \to M$ so that $TM \oplus E$ is trivial. (The bundle E will end up being the normal bundle to the immersion). See [**H2**] (for $m < n$) and [**H3**] (for $n = m$). We also did a number of miscellaneous things. We looked at a one to one immersion which is not an imbedding, a line with irrational slope in the torus. Bredon calls this a submanifold, but we will not. For us, submanifolds are always embedded. We defined isotopy of submanifolds. We defined manifolds with boundary. A manifold M with boundary has an interior $\mathrm{Int}M$ which is a manifold without boundary, and a boundary ∂M which is a manifold without boundary of one lower dimension. We talked briefly about manifolds with corners (which arise for example when you take the Cartesian product of two manifolds), but quickly dismissed them. The point is that there is a standard way to smooth the corner to give them the structure of a manifold with boundary in a unique way, up to diffeomorphism. Next we showed that any continuous map $f\colon M \to N$ can be approximated by a smooth map, by proving the relative local version and invoking the standard argument. A similar argument shows that any C^k map (k times differentiable) can be approximated by a C^∞ map, so that all k derivatives are also approximated, but I don't expect us to make use of that. References [**BJ**, 88-97,129-131], [**Br**, 71], [**H**, 29-30,41-49].

9/26 We talked about Sard's Theorem which says if $f\colon M \to N$ is smooth, then the critical values of f have measure zero, and in particular the regular values are dense. So if $\dim M < \dim N$, $f(M)$ is nowhere dense. We used Sard's theorem to prove that if $f\colon M \to \mathbb{R}^n \times \mathbb{R}$ is an immersion and $n \geq 2 \dim M$ then for almost all linear projections $\pi_v\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\pi_v f$ is also an immersion. Moreover, if f is one to one and $n \geq 2 \dim M + 1$ then $\pi_v f$ is a one to one immersion for almost all $v \in \mathbb{R}^n$. ($\pi_v(x, t) = x - tv$). ("Almost all" means "all but a set of measure zero".) References [**BJ**, 56-61], [**H**, 24-27,67-72], [**Mi1**, 16-19], [**Br**, 531-534,90-92].

9/29 We show that if $f\colon M \to N$ and $m = \dim M$, $n = \dim N$ then:

a) f can be approximated by an immersion g if $n \geq 2m$.

b) f can be approximated by a one to one immersion g if $n \geq 2m+1$.

c) If f is proper, then f can be approximated by an embedding g if $n \geq 2m+1$.

If f is already an immersion or embedding an a neighborhood of a closed set A we may also conclude $f|_A = g|_A$. If f is smooth, we may take g to be a smooth approximation, we may specify that all partial derivatives of order $\leq k$ are close also. The proof follows from the local relative version by the usual argument. For the local version, essentially take the graph of f which immerses or embeds $\mathbb{R}^m$ in $\mathbb{R}^n \times \mathbb{R}^m$ and use the 9/26 argument above to do generic projections one dimension at a time until we immerse or 1-1 immerse in $\mathbb{R}^n$. If we always choose projections π_v with v very small the result will closely approximate f. Note f is proper means $f^{-1}(K)$ is compact for all compact K. If f is proper then the approximation g will also be proper, but a proper one to one immersion is an embedding. (In general a proper continuous one to one onto map between locally compact Hausdorff spaces is a homeomorphism, which can be seen by extending to the one point compactification.) Since any smooth manifold admits a proper smooth map to $\mathbb{R}$, we then see that any smooth manifold of dimension m embeds in $\mathbb{R}^{2m+1}$. In fact, an argument of Whitney shows it embeds in $\mathbb{R}^{2m}$, but as simple examples show, it is not possible to approximate any map to $\mathbb{R}^{2m}$ by an embedding as one can do in higher dimensions (for proper maps anyway). We also showed that if $n = 2m$ then f can be approximated by an immersion g which is at most two to one, and moreover if $g(x) = g(y)$, $x \neq y$, then $dg_x(TM_x) + dg_y(TM_y) = TN_{g(x)}$ so the image of the immersion looks locally there like two complementary m planes in $\mathbb{R}^{2m}$ (if is proper, otherwise it could get pretty messy). References [**BJ**, 65-72], [**H**, 23-27,53-55], [**Br**, 91-92].

10/1 Often proofs are easier if we use the fact a manifold can be embedded in some $\mathbb{R}^k$. For example, let us show that C^1 close embeddings are isotopic. Suppose $f_i\colon M \to N$ are close embeddings. Embed N in $\mathbb{R}^k$. By the tubular neighborhood theorem there is a smooth retraction $r\colon U \to N$ for some neighborhood U of M in $\mathbb{R}^k$. (In fact for a canonical choice of tubular neighborhood, we could take r to be the closest point map.) The map $H(x,t) = r(tf_1(x) + (1-t)f_0(x))$ is an isotopy from f_0 to f_1 if f_0 and f_1 are C^1 close enough. Another corollary of the embedding theorem is that there are no knots in $\mathbb{R}^n$ for $n > 3$, any embedding $S^1 \to \mathbb{R}^n$ is isotopic to the standard $S^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \mathbb{R}^n$. In fact we show that any two homotopic embeddings $f_i\colon M^m \to N^n$ are isotopic if $n \geq 2m+2$ and M is compact. (It's also true if M is noncompact and the embeddings and homotopy are proper). This is easy for $n > 2m+2$, approximate the homotopy by a smooth homotopy $H\colon M \times I \to N$ so $H(x,i) = f_i(x)$, $i = 0,1$. Now approximate H by an embedding $G\colon M \times I \to N$. Then $H|_{M\times i}$ is isotopic to $G|_{M\times i}$, for $i = 0,1$ and G provides an isotopy from $G|_{M\times 0}$ to $G|_{M\times 1}$. The argument for $n = 2m+2$ is more delicate, and well worth doing. I give two arguments, each of which illustrates a useful concept we will investigate later. One argument considers the map $h\colon M \times I \to N \times I$ given by, $h(x,t) = (H(x,t),t)$, and C^1 approximates it by an embedding $g\colon M \times I \to N \times I$ so $g|_{M\times i} = h|_{M\times i}$ for $i = 0,1$. If $\pi\colon N \times I \to I$ and $\rho\colon N \times I \to N$ are projections, then $\pi g(x,t)$ is a C^1 approximation of the map $(x,t) \mapsto t$. Hence one can argue that there is a diffeomorphism $f\colon M \times I \to M \times I$ so that $\pi g f(x,t) = t$ (for example $f(x,t) = (x, \alpha(x,t))$ where $\alpha(x,t)$ is the unique s so $\pi g(x,s) = t$). Then gf embeds $M \times t$ to $N \times t$ so $\rho g f\colon M \times I \to N$ is an isotopy from f_0 to f_1. A second argument notes that as on 9/29 we may approximate H by an immersion G which at worst has transverse double points. These double points are all isolated so there are a finite number of $(x_i, t_i) \in M \times I$ so that G embeds their complement. By perturbing G sligfhtly we may assume all t_i are distinct. Then G gives an isotopy since G embeds each $M \times t$. This last argument uses the important notion of transversality which is our next topic. Two embedded submanifolds $M, P \subset N$ are transverse if whenever $q \in M \cap P$ then $TM_q + TP_q = TN_q$. This imples that we can find local coordinates in which M and P are both linear subspaces which intersect with minimal dimension $m+p-n$. Consequently $M \cap P$ is a submanifold. More generally if $f\colon M \to N$ and $g\colon P \to N$ are smooth maps we say f and g are transverse if whenever $f(x) = g(y)$ we have $df_x(TM_x) + dg_y(TP_y) = TN_{f(x)}$. References [**BJ**, 51], [**H**, 22], [**Br**, 84-86].

10/3 If f and g are smooth we can approximate g by a map g' so that f and g' are transverse. For example, if M and P are embedded submanifolds, we may isotop P to be transverse to M. To prove map transversality, consider the local form where $N = \mathbb{R}^n$. We have a smooth map $h\colon M \times P \to \mathbb{R}^n$ given by $h(x,y) = f(x) - g(y)$. If v is a regular value of h then f is transverse to $g+v$ (i.e., g followed by translation by v). The proof for general N is by the usual process of piecing together the relative version of this local result, or by using the Baire category theorem (since the set of transverse g's is open in the

C^1 topology). If M and P are transverse embedded submanifolds of N, then $M \cap P$ is an embedded submanifold of N with dimension $m+p-n$. If $f: M \to N$ is transverse to an embedded submanifold P, then $f^{-1}(P)$ is an embedded submanifold of M with dimension $m+p-n$. References [**BJ**, 145-151], [**H**, 74-80], [**Br**, 114-118].

10/6 We want to show that if f and g are transverse and g' is C^1 close to g, then somehow the situation has not changed. In particular, if f and g are embeddings, then $f(M) \cap g(P)$ is isotopic to $f(M) \cap g'(P)$. If f is an embedding, then $g^{-1}(f(M))$ is isotopic in P to $g'^{-1}(f(M))$. If g is an embedding, then $f^{-1}(g(P))$ is isotopic in M to $f^{-1}(g'(P))$. (In all cases assume M and P are compact and without boundary to avoid more complicated statements). To show this we will first develop the technique of integrating vector fields. A smooth vector field v on a manifold M is a smooth section $v: M \to T(M)$. By existence and uniqueness of ODEs there is a neighborhood U of $M \times 0$ in $M \times \mathbb{R}$ and a $\varphi: U \to M$ so that $\varphi(x,0) = x$ and $d\varphi(x,t)/dt = v(\varphi(x,t))$. Uniqueness of ODE solutions guarantees that in fact $\varphi(\varphi(x,t),s) = \varphi(x,s+t)$. If M is compact without boundary (or v has compact support) we can take $U = M \times \mathbb{R}$, but otherwise is is possible that $\varphi(x,t)$ "goes to infinity" as t approaches some finite limit a. We call φ the flow associated to v, or say we integrate v to get the flow φ. Some authors may call φ a partial flow if $U \neq M \times \mathbb{R}$. But in any case, we can take U as big as possible, so $U \cap x \times \mathbb{R} = (a_x, b_x)$ for some interval so $\lim_{t \to a_x} \varphi(x,t)$ and $\lim_{t \to b_x} \varphi(x,t)$ do not exist. Note $a_{\varphi(x,t_0)} = a_x - t_0$ and $b_{\varphi(x,t_0)} = b_x - t_0$. Note that if $M \times t_0 \subset U$ then the map $x \mapsto \varphi(x,t_0)$ is a diffeomorphism with inverse $x \mapsto \varphi(x,-t_0)$. Note also that this diffeomorphism is isotopic to the identity by the isotopy $H_t(x) = \varphi(x, tt_0)$. It turns out vector fields are a great way to construct diffeomorphisms and isotopies. If M is a manifold with boundary, there is a slight variation in the above discussion. You allow U not to be open and then possibly $U \cap x \times \mathbb{R} = [a_x, b_x)$ or $(a_x, b_x]$ or $[a_x, b_x]$. The first case, for example, would correspond to $\varphi(x, a_x) \in \partial M$. References [**BJ**, 74-85], [**H**, 149-151], [**Br**, 86-88]

Suppose Q is a compact embedded submanifold of $M \times I$, $\partial M = \phi$, $Q \cap (M \times \{0,1\}) = \partial Q$. Suppose also that the tangent space of Q is never horizontal, $TQ_(x,t) \not\subset TM_x \times 0$. Another way of saying this is that the map $\pi: Q \to I$ given by $\pi(x,t) = t$ has no critical points. Define $Q_t \subset M$ by $Q_t \times t = Q \cap M \times t$. Then Q_0 is isotopic to Q_1 in M. To prove this note there is a smooth vector field v on Q so $d\pi(v) = 1$. (We only need construct v locally and then piece together with a partition of unity.) Integrate v to get a flow φ. Then $d\pi\varphi(x,t)/dt = d\pi(v) = 1$, so $\pi\varphi(x,t) = \pi(x) + t$. We have an isotopy $H_t: Q_0 \to M$ defined by $H_t(x) = \rho\varphi((x,0),t)$, where $\rho: M \times I \to M$ is projection. Note $H_t(Q_0) = Q_t$ and $H_t^{-1} : Q_t \to Q_0$ is $H_t^{-1}(x) = \rho\varphi((x,t),-t)$ so each H_t is in fact an embedding. Note that in fact we get a bit more. We can extend v to a vector field on $M \times I$ so that $v = (0,1)$ outside a compact set. Then integrating v gives an isotopy of $H_t: M \to M$ so each each H_t is a diffeomorphism (sometimes called a diffeotopy or more commonly an ambient isotopy). So not only is Q_0 isotopic to Q_1, but there is an ambient isotopy of M which takes Q_0 to Q_1. This proves the isotopy extension theorem, that any isotopy of embeddings of a compact manifold can be extended to an ambient isotopy [**BJ**, 91], [**H**, 180].

10/8 Suppose $g: P \to N$ and that $f: M \to N$ is an embedding and f and g are transverse. Assume that either P is compact, or that M is compact and g is proper. Also M, P, and N have empty boundary. Then if g' is C^1 close to g, we want to show that $g^{-1}(f(M))$ is isotopic to $g'^{-1}(f(M))$. First we show there is a smooth homotopy $G: P \times I \to N$ from g to g' which is C^1 close to the map $(x,t) \mapsto g(x)$. (Use the construction given at the beginning of 10/1 for example). Then G is transverse to f, so $Q = G^{-1}(f(M))$ is a submanifold of $P \times I$. Now $TQ = dG^{-1}(df(TM)) \approx dg^{-1}(df(TM)) \times TI$ so TQ is never horizontal. So by the theorem proven on 10/6, $g^{-1}(f(M))$ is isotopic to $g'^{-1}(f(M))$. In fact there is an ambient isotopy of P which takes $g^{-1}(f(M))$ to $g'^{-1}(f(M))$. Similarly, suppose $g: P \to N$ is an embedding, $f: M \to N$, and f and g are transverse. Assume that either M is compact, or that P is compact and f is proper. Also M, P, and N have empty boundary. Then if g' is C^1 close to g, we want to show that $f^{-1}(g(P))$ is isotopic to $f^{-1}(g'(P))$. As on 10/1, we have an isotopy $G: P \times I \to N$ from g to g' which is C^1 close to the map $(x,t) \mapsto g(x)$. Let $G': P \times I \to N \times I$ be the map $G'(x,t) = (G(x,t),t)$. Then $dG' \approx dg \times id$ so G' is transverse to the map $f \times id: M \times I \to N \times I$. Let $Q = (f \times id)^{-1}(G'(P \times I))$. Then

$$TQ = (df \times id)^{-1}(dG'(T(P \times I))) \approx (df \times id)^{-1}(dg(TP) \times TI) = df^{-1}(dg(TP)) \times TI$$

so TQ is never horizontal. So $f^{-1}(g(P))$ is isotopic to $f^{-1}(g'(P))$. At the end we briefly looked at uniqueness of tubular neighborhoods and existence and uniqueness of collars on ∂M. A collar is an embedding $c: \partial M \times [0,1) \to M$ so that $c(x,0) = x$. Collars exist (they are really just a type of tubular neighborhood so the tubular neighborhood existence proof works). Also collars and tubular neighborhoods are unique up to isotopy. Any two collars are isotopic. For any two tubular neighborhoods $\varphi, \psi: E \to N$, there is an isomorphism $h: E \to E$ of the normal bundle so that φ and ψh are isotopic. In fact h is the restriction of $d\psi^{-1}\varphi$ to the zero section (note $TE|_M$ is isomorphic to E). We proved the local form of this where $M = \mathbb{R}^m$, $E = \mathbb{R}^m \times \mathbb{R}^k$, $\psi^{-1}\varphi(x,y) = (f(x,y), g(x,y))$. We then use the isotopy $h_t(x,y) = (f(x,ty), g(x,ty)/t)$. This has a smooth limit when $t = 0$ (c.f., [**BJ**, 101,127], [**H**, 112]). Note $h_0(x,y) = (x, A_x(y))$ where A_x is a nonsingular $k \times k$ matrix depending on x. (Its i,j component is $\partial g_i/\partial y_j(x,0)$.)

10/10 We looked last time at uniqueness of open tubular neighborhoods of $M \subset N$, but the isotopy might not be able to be covered by an ambient isotopy of N. For example, $\mathbb{R}^2$ and the open disc of radius 1 are both tubular neighborhoods of the origin in $\mathbb{R}^2$. They are isotopic as embeddings but are not ambiently isotopic. But if you put a Reimannian metric on the bundle E and look at the image of the unit disc bundle (all vectors of length ≤ 1) you get a closed tubular neighborhood. Then if M is compact, there is an ambient isotopy of N which takes any closed tubular neighborhood to any other. To see this, suppose $\varphi, \psi: E \to N$ are two tubular neighborhoods. We know there is an isotopy G_t from φ to ψh for some bundle isomorphism $h: E \to E$.. Let $S \subset E$ be the unit sphere bundle, the set of vectors of length 1. There is a linear isotopy in E which takes $h(S)$ to S (just take $(v,t) \mapsto v/(1-t+t||v||)$) so composing this with G we get an isotopy of embeddings G'_t so $G'_0(S) = \varphi(S)$ and $G'_1(S) = \psi(S)$. Now S is compact, so by the isotopy extension theorem there is an ambient isotopy $H_t: N \to N$ so that $H_t\varphi(x) = G'_t(x)$ for all $x \in S$, and $H_0 =$ identity. So $H_1(\varphi(S)) = \psi(S)$. But then note that $H_1(\varphi(D)) = \psi(D)$ if D is the unit disc bundle. So closed tubular neighborhoods of compact manifolds are unique up to ambient isotopy. Likewise, closed collars $c: \partial M \times [0,1] \to M$ are unique up to ambient isotopy if ∂M is compact. We then started talking about differential forms.

10/13 James Crispino lectured on an introduction to differential forms, covering roughly page 260-264 of Bredon.

10/15 James proves Stokes' Theorem, roughly 265-269 of Bredon.

10/17 We looked at orientation, showing that an n dimensional manifold is orientable if and only if it has a nowhere zero n form. We then looked at a construction which gives a Poincare dual of the first Stiefel-Whitney class. Take a section τ of $\Lambda^n(TN^*)$ transverse to the zero section. Then $\tau^{-1}(0)$ is an $n-1$ dimensional submanifold W of N. Transverse intersection with W gives a homomorphism $\pi_1(N, x_0) \to \mathbb{Z}/2\mathbb{Z}$ (count the number of points of intersection of a curve with W). Curves mapping to 1 are curves along which N is not orientable. We can look at this as an element $\omega_1 \in H^1(N; \mathbb{Z}/2\mathbb{Z})$ called the first Stiefel-Whitney class. But the submanifold W is what is behind it. We then looked at the oriented bordism group of a space $\Omega_n(X)$. Elements are equivalence classes of continuous maps $f: N \to X$ where N is a compact oriented n dimensional manifold without boundary. Two such maps $f_i: N_i \to X$, $i = 0,1$ are equivalent if there is a compact oriented manifold W so that ∂W is the disjoint union of M_0 (with the given orientation) and M_1 (with the opposite orientation).

10/20 More on $\Omega_n(X)$. The group operation is disjoint union. The zero is either the empty manifold (if you are comfortable with the empty manifold having any dimension) or a constant map of the sphere S^n to X. This is a natural invariant of a space (called a functor), for example a map $g: X \to Y$ induces a map $g_*: \Omega_n(X) \to \Omega_n(Y)$ via composition. Homotopic g's induce the same g_*'s. We can take the direct sum of $\Omega_n(X)$ for all n and get a graded group $\Omega_*(X)$. If X is a smooth manifold, and ω is a closed n form on X (closed means $d\omega = 0$), then Stokes' theorem implies that $\int_N f^*(\omega)$ is independent of the bordism class of f, so ω gives a homomorphism from $\Omega_n(X)$ to $\mathbb{R}$. (Note any bordism class contains a smooth representative since we may approximate any map by a smooth map homotopic to the original). Note also that if ω is exact (which means $\omega = d\tau$ for some $n-1$ form τ) then $\int_N f^*(\omega) = \int_N df^*(\tau) = \int_{\partial N} f^*(\tau) = 0$ since ∂N is empty. So if we denote $H^n_\Omega(X) =$ closed n forms on X modulo the subspace of exact n forms, we have a homomorphism $H^n_\Omega(X) \to Hom(\Omega_n(X), \mathbb{R})$.

Unfortunately $\Omega_n(X)$ is difficult to compute, even if X is a point. We're better off with an invariant which is more complicated to define, but easier to compute and work with. Such an invariant is homology. You can think of homology as like oriented bordism, except that you use simplicial complexes instead of manifolds, and you can jazz things up by picking a coefficient group and assigning a group element to each n simplex in the simplical complex. the end result is that for any space X and abelian group G you get a natural graded group $H_*(X;G)$. If $g\colon X \to Y$ there is an induced map $g_*\colon H_*(X;G) \to H_*(Y;G)$. Some results, $H_0(X;G) = G^\ell$ where ℓ is the number of path components of X. If X is path connected then $H_1(X;G) = \pi' \otimes G$ where π' is the abelianization of $\pi_1(X, x_0)$. For now we wil focus on $G = \mathbb{R}$, then in fact each $H_n(X;\mathbb{R})$ is a real vector space. Then DeRham's theorem says the map $H^n_\Omega(X) \to Hom(H_n(X;\mathbb{R}),\mathbb{R}) = H_n(X;\mathbb{R})^*$ is an isomorphism for any smooth manifold X.

10/22 We outlined a proof of DeRham's theorem, that the integration map

$$H^n_\Omega(X) \to Hom(H_k(X;\mathbb{R}),\mathbb{R}) = H_k(X;\mathbb{R})^*$$

is an isomorphism for any k and smooth manifold X. The proof has these steps.
a) Show if DR is true for A, B, and $A \cap B$, it is true for $A \cup B$, where A and B are open.
b) Show DR is true for $X = \mathbb{R}^n$.
c) Since $S^k \times \mathbb{R}^{n-k} = A \cup B$ where A and B are diffeomorphic to $\mathbb{R}^n$ and $A \cap B$ is diffeomorphic to $S^{k-1} \times \mathbb{R}^{n-k+1}$, a) and b) and induction on k show that DR is true for $S^k \times \mathbb{R}^{n-k}$, (where S^k is the k sphere).
d) Using future work (Morse theory) show any n manifold X is the union of open sets A_i so that each A_i is diffeomorphic to $\mathbb{R}^n$, and each $A_i \cap (A_1 \cup A_2 \cup \ldots \cup A_{i-1})$ is diffeomorphic to $S^k \times \mathbb{R}^{n-k}$ for some k.
e) Then a), b), c), d) imply DR for all compact manifolds. The noncompact case follows with a little more argument, using, as usual, compact bands going off to infinity.
A similar argument is given in [**Br**, 286-291].

10/24 We looked at more of the proof of deRham's theorem. In particular we showed the sequence

$$0 \to \Omega^k(A \cup B) \to \Omega^k(A) \oplus \Omega^k(B) \to \Omega^k(A \cap B) \to 0$$

is exact which leads to a long exact sequence

$$\cdots \to H^k_\Omega(A \cup B) \to H^k_\Omega(A) \oplus H^k_\Omega(B) \to H^k_\Omega(A \cap B) \to H^{k+1}_\Omega(A \cup B) \to \cdots$$

For amusement, we calculated the map $H^k_\Omega(A \cap B) \to H^{k+1}_\Omega(A \cup B)$ which is $\omega \mapsto d\psi \wedge \omega$ where $\psi\colon A \cup B \to [0,1]$ is any smooth map which is 0 on $A - B$ and 1 on $B - A$. Let $H^k(X;\mathbb{R}) = H_k(X;\mathbb{R})^*$, called the real cohomology of X. There is a similar long exact sequence for cohomology, so by a standard algebraic topology argument (called the 5 lemma) comparing these to sequences shows that if DeRham's theorem is true for A, B. and $A \cap B$ then it is true for $A \cup B$. Next we showed that DeRham's theorem is true for $\mathbb{R}^n$, assuming a result we will prove 10/26, that homotopic maps induce the same map on DeRham cohomology.

10/27 If $F\colon M \times [a,b] \to N$ is a smooth homotopy, define $\varphi\colon\ \Omega^k(N) \to\ \Omega^{k-1}(M)$ by

$$\varphi(\omega)(v_1,\ldots,v_{k-1}) = \int_a^b F^*(\omega)(\partial/\partial t, v_1,\ldots,v_{k-1})\,dt$$

where t is the coordinate in the $[a,b]$ direction. This map is linear. I claim that $d\varphi + \varphi d = f_b^* - f_a^*$ where $f_t(x) = F(x,t)$. It suffices to prove this locally. (There are a few words to say about being local in N, since the homotopy could go all over the place. But given a small compact neighborhood K of a point in M, we may subdivide $[a,b]$ into smaller intervals $[t_i, t_{i+1}]$, on each of which $F(K \times [t_i, t_{i+1}])$ remains in a chart. Adding up the result for each smaller interval, we get the desired result.) So we may suppose that M and N are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

Let us temporarily introduce some notation. Let F_i be the i-th coordinate of F. If I is a k-tuple of integers $I = (i_1, i_2, \ldots, i_k)$ let $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, $dF_I = dF_{i_1} \wedge dF_{i_2} \wedge \ldots \wedge dF_{i_k}$, and let $\partial/\partial x_I$ be the ordered k-tuple of vectors $(\partial/\partial x_{i_1}, \ldots, \partial/\partial x_{i_k})$. Let I_j be the $k-1$-tuple obtained by deleting i_j. Note that $F^*(dx_J) = dF_J$. Also note that if ω is a k form then $d\omega(\partial/\partial x_I) = \sum_{j=1}^{k+1}(-1)^{j-1}\partial/\partial x_{i_j}(\omega(\partial/\partial x_{I_j}))$. Now suppose $\omega \in \Omega^k(N)$. We will show that $(d\varphi(\omega) + \varphi(d\omega))(\partial/\partial x_I) = (f_b^*\omega - f_a^*\omega)(\partial/\partial x_I)$ which by linearity implies that $d\varphi + \varphi d = f_b^* - f_a^*$. We have

$$d\varphi(\omega)(\partial/\partial x_I) = \sum_{j=1}^{k}(-1)^{j-1}\partial/\partial x_{i_j}(\varphi(\omega)(\partial/\partial x_{I_j}))$$

$$= \int_a^b \sum_{j=1}^{k}(-1)^{j-1}\partial/\partial x_{i_j}(F^*(\omega)(\partial/\partial t, \partial/\partial x_{I_j}))\,dt$$

We also have

$$\varphi(d\omega)(\partial/\partial x_I) = \int_a^b dF^*(\omega)(\partial/\partial t, \partial/\partial x_I)\,dt$$

$$= \int_a^b \partial/\partial t(F^*(\omega)(\partial/\partial x_I)) + \sum_{j=1}^{k}(-1)^{j}\partial/\partial x_{i_j}(F^*(\omega)(\partial/\partial t, \partial/\partial x_{I_j}))\,dt$$

Thus

$$(d\varphi(\omega) + \varphi(d\omega))(\partial/\partial x_I) = \int_a^b \partial/\partial t(F^*(\omega)(\partial/\partial x_I))\,dt$$

$$= F^*(\omega)(\partial/\partial x_I)|_a^b = (f_b^*\omega - f_a^*\omega)(\partial/\partial x_I)$$

So we have shown $d\varphi + \varphi d = f_b^* - f_a^*$. Thus if ω is a closed form representing a cohomology class in $H^*_\Omega(N)$, we have $f_b^*(\omega) - f_a^*(\omega) = d\varphi(\omega) + \varphi(d\omega) = d\varphi(\omega)$ and thus $f_b^*(\omega)$ and $f_a^*(\omega)$ represent the same cohomologty class. So homotopic maps induce the same map on DeRham cohomology (since any homotopy may be approximated by a smooth homotopy).
As an exercise, consider the map $F(x,t) = tx$ from $\mathbb{R}^n \times [0,1]$ to $\mathbb{R}^n$. Show that φ is the map ϕ given by Bredon on page 288.

10/29 References for the DeRham isomorphism being a ring homomorphism are [**BT**, 175] and [**W**, 214] but it I forgo explaining this in class due to the machinery needed. Today we started characteristic classes. Let $G_{n,k}$ be the Grassmanian of k planes in n space. There is a canonical rank k vector bundle η over $G_{n,k}$ where the inverse image of a point $Q \in G_{n,k}$ is the vector space Q. If X is paracompact then the set of rank bundles over X is in one to one correspondence with the set of homotopy classes of X to $G_{\infty,k}$, [**MS**, 65]. The correspondence is that a map $f\colon X \to G_{n,k}$ corresponds to the induced bundle $f^*(\eta)$. We proved this if X is a CW complex with a finite number of cells, then bundles over X correspond to homotopy classes of maps to $G_{n,k}$ if $n > k + \dim X$. The crucial Lemma to prove this is that if $V_{n,k}$ is the Stiefel manifold of orthonormal sets of k vectors in $\mathbb{R}^n$, then any map of an m sphere to $V_{n,k}$ is null homotopic, if $m < n-k$. Then we can construct the classifying map f cell by cell.

10/31 We looked at the decomposition of the Grassmanian $G_{n,k}$ into Schubert cells, each cell consisting of row spaces of matrices whose row reduced echelon form has pivots in a particular place [**MS**, 73-81]. This represents $G_{n,k}$ as a CW complex, and in fact each of these cells is an independent generator of $H^*(G_{\infty,k}, Z/2Z)$. I stated without proof that as a ring, $H^*(G_{\infty,k}, Z/2Z)$ is commutative and freely generated by $w_i \in H^i(G_{\infty,k}, Z/2Z)$, $i = 0, \ldots, k$ [**MS**, 82-88]. Thus we can get characteristic classes of a bundle $\omega_i(\xi) = h^*(w_i) \in H^i(X; Z/2Z)$ if ξ is induced by a map $h\colon X \to G_{\infty,k}$. These are called the Stiefel-Whitney classes. If we let $\omega_*(\xi) = \omega_0(\xi) + \omega_1(\xi) + \ldots$ then $\omega_*(\xi \oplus \zeta) = \omega_*(\xi)\omega_*(\zeta)$ [**MS**, 92-93]. Since $\omega_0(\xi) = 1$ always, $\omega_*(\xi)$ is invertible, so $1/\omega_*(\xi)$ makes sense, e.g., $1/(1+a) = 1 - a + a^2 - a^3 + \cdots$ If M is a manifold, then we define $\omega_*(M) = \omega_*(TM)$. If $M \subset \mathbb{R}^n$ is a submanifold and ν is its normal bundle, then $TM \oplus \nu$ is trivial so $\omega_*(\nu) = 1/\omega_*(M)$. We computed $\omega_*(S^n) = 1$ and $\omega_*(\mathbb{RP}^n) = (1+a)^{n+1}$ where $a \in H^1(\mathbb{RP}^n; Z/2Z) = Z/2Z$ is the generator [**MS**, 45-46].

11/3 We start with another point of view of the Stiefel-Whitney classes, as obstructions to framing. Suppose we have a rank k vector bundle ξ whose base X is a CW complex. In particular X has closed subsets $X_0 \subset X_1 \subset X_2 \subset \cdots$ so that X_i is obtained from X_{i-1} by gluing on some cells of dimension i and $X = \bigcup_i X_i$. An ℓ framing of the bundle is a choice of ℓ linearly independent sections (i.e., they are linearly independent at each point of X). We saw on 10/29 that any map of the m sphere to $V_{k,\ell}$ is null homotopic if $m < k-\ell$. Consequently we can always construct an ℓ framing of $\xi|_{X_{k-\ell}}$. Our 10/29 proof also showed that when trying to extend this framing to $\xi|_{X_{k-\ell+1}}$ then for each $k-\ell+1$ we end up with a mapping of its boundary $k-\ell$ sphere to a $k-\ell$ sphere which we would like to be null homotopic, but might not be. We get a cohomology class in $H^{k-\ell+1}(X; Z/2Z)$ by mapping each $k-\ell+1$ cell to the mod 2 degree of this map of spheres. This is $\omega_{k-\ell+1}(\xi)$ [**MS**, 139-143]. In particular, if there is an ℓ framing of $\xi|_{X_{k-\ell+1}}$ it is necessary that $\omega_{k-\ell+1} = 0$. For example, $\omega_1(\xi)$ is the obstruction to getting a k-framing over X_1, which is equivalent to an orientation of ξ. Also $\omega_k(\xi)$ is an obstruction to finding a 1 framing on X_k, i.e., a nowhere zero section on X_k. For example, if M is a manifold of dimension m then $\omega_m(M)$ is an obstruction to finding a nowhere zero vector field on M.

If ξ is orientable then we can lift $\omega_k(\xi)$ to an integral cohomology class since our obstruction is a map of S^{k-1} (the boundary of an attached k cell) to S^{k-1} (the unit sphere in the vector bundle fibers). Since the bundle is orientable, we have a preferred orientation of this destination S^{k-1} and hence a well defined integer degree of the map. hence we get a cohomology class $e(\xi) \in H^k(X; Z)$ with Z coefficients called the Euler class of the bundle. If M is a k dimensional oriented manifold without boundary, then $e(TM)([M]) = \chi(M)$ where $[M] \in H_k(M; Z)$ is the orientation class and $\chi(M)$ is the Euler characteristic of M [**MS**, 130].

If we let $\tilde{G}_{n,k}$ be the two-fold orientation cover of $G_{n,k}$, then orientable bundles are classified by maps to $\tilde{G}_{\infty,k}$. This because a map $h\colon X \to G_{n,k}$ lifts to a map to $\tilde{G}_{n,k}$ if and only if the induced bundle is orientable.

11/5 There are more characteristic classes of oriented bundles but to define them we first look at characteristic classes of complex bundles. Just as in the real case, complex bundles over a paracompact base space X are in one to one correspondence with homotopy classes of maps of X to $G_{\infty,k}(\mathbb{C})$ where $G_{n,k}(\mathbb{C})$ is the Grassmannian of k dimensional complex linear subspaces of $\mathbb{C}^n$. We can again decompose $G_{n,k}(\mathbb{C})$ into Schubert cells. But now since all cells have even real dimension, the boundary maps in the cohomology chain complex are all trivial so $H^i(G_{n,k}(\mathbb{C}); Z)$ is 0 for i odd and is Z^{ℓ_i} for i even where ℓ_i is the number of Schubert cells of complex dimension $i/2$. Similarly to the real case, this cohomology ring is freely generated by classes $c_i \in H^{2i}(G_{n,k}(\mathbb{C}); Z)$. If a complex bundle ξ is represented by a map $h\colon X \to G_{n,k}(\mathbb{C})$ then the Chern classes $c_i(\xi)$ are defined by $c_i(\xi) = h^*(c_i)$. We have the usual properties $c_*(\xi \oplus \zeta) = c_*(\xi)c_*(\zeta)$, $c_0(\xi) = 1$, and $c_*(f^*(\xi)) = f^*(c_*(\xi))$.

Given any real vector bundle ξ, we may complexify it to get a complex bundle $\xi \otimes \mathbb{C}$ as follows. Take $\xi \oplus \xi$ and define $J\colon \xi \oplus \xi \to \xi \oplus \xi$ by $J(x \oplus y) = (-y) \oplus x$. The points of $\xi \otimes \mathbb{C}$ are $\xi \oplus \xi$ and J gives multiplication by $\sqrt{-1}$. It turns out that the odd Chern classes of $\xi \otimes \mathbb{C}$ all have order 2. We define the Pontryagin classes of ξ by $p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(X; Z)$.

The ring $H^*(\tilde{G}_{\infty,k}; Z)$ has a bunch of complicated 2 torsion, but if we kill the 2 torsion, for example by switching to real coefficients then the ring $H^*(\tilde{G}_{\infty,k}; \mathbb{R})$ is freely generated by the Pontryagin classes $p_0, p_1, \ldots p_{(k-1)/2}$ if k is odd and by the Pontryagin classes $p_0, p_1, \ldots p_{k/2-1}$ and the Euler class e if k is even, and moreover $e^2 = p_{k/2}$. Thus the Pontryagin classes and Euler class are the $\mathbb{R}$ characteristic classes of orientable bundles (and in fact the G characteristic classes for any integral domain G without 2 torsion) [**MS**, 179].

Nicholas Long started talking about Morse theory. He defined a nondegenerate critical point of $f\colon M \to \mathbb{R}$, a point p where $df_p = 0$ and in local coordinates the Hessian matrix of second partials $H = (\partial^2 f/\partial x_i \partial x_j)$ is nonsingular. This matrix is symmetric, hence diagonalizable with all real eigenvalues. The index of the critical point is the number of negative eigenvalues. Note that the second order taylor series of f is $f(x) \approx f(0) + x^T Hx/2$ if we choose our coordinates so the critical point p is at 0. In fact we can extend H to be a matrix valued function $H(x)$ so that we have equality $f(x) = f(0) + x^T H(x)x/2$, or we could write this as $f(x) = f(0) + \sum_{i=1}^m \sum_{j=1}^m x_i x_j h_{ij}(x)$ with $h_{ij} = h_{ji}$. One can diagonalize a quadratic form by completing the square, doing this with the above not quite quadratic form leads to a change of variables so that $f(y) = f(0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^m x_i^2$ where λ is the index of the critical point. (Step one is to let $z_1 = x_1 + x_2h_{12}(x)/h_{11}(x) + \cdots + x_mh_{1m}(x)/h_{11}(x)$, then $h_{11}(x)z_1^2$ absorbs all the x_1x_j terms. Now let $y_1 = \sqrt{|h_{11}(x)|}z_1$, then $\pm y_1^2$ absorbs all the x_1x_j terms.) So the whole point is that at a nondegenerate critical point, the function f has a particularly simple form, a sum of $\pm$ squares of the variables. (Note, actually you can do this process at a degenerate critical point too in a more careful way and get $f(y) = f(0) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{\mu} x_i^2 + g(x_{\mu+1}, \ldots, x_m)$ where $m - \mu$ is the number of 0 eigenvalues of H and g is totally degenerate, its second order Taylor series is 0.)

11/7 A Morse function is one whose critical points are all nondegenerate. Any function can be approximated by a Morse function, in fact Morse functions are open and dense in the space of all functions. You might be amused to fill in the details to the following proof of denseness. Any function $M \to \mathbb{R}$ may be approximated by a smooth function f, which by embedding M in $\mathbb{R}^n$ and taking the graph of f, we may assume is a coordinate function for some fixed embedding of M in Euclidean space. So it suffices to show that if $M \subset \mathbb{R}^n$ is a submanifold, then for almost all linear functions $L\colon \mathbb{R}^n \to \mathbb{R}$ the restriction $L|_M$ is Morse. If $v \in \mathbb{R}^n$, let $L_v(x) = v \cdot x$, then any linear function is some L_v. Let

$$X = \{(x, v) \in M \times \mathbb{R}^n \mid x \text{ is a critical point of } L_v|_M\} = \{(x, v) \mid v \perp TM_x\}$$

Then X is an n dimensional manifold (in fact the normal bundle of M). Let $h\colon X \to \mathbb{R}^n$ be the map $h(x, v) = v$. Then a bit of calculation shows that critical points of h are exactly points (x, v) where $L_v|_M$ has a degenerate critical point at x. Thus if v is a regular value of h then $L_v|_M$ is a Morse function. A similar but easier calculation shows that if $f\colon \mathbb{R}^m \to \mathbb{R}$ is smooth, then for almost all $v \in \mathbb{R}^m$ the function $x \mapsto f(x) + v \cdot x$ is Morse. Thus maps which are Morse in a given compact piece of a coordinate chart are open and dense in the space of all maps. The Baire category theorem then gives density. References [**Mi2**, 4-8,32-38], [**H**, 143-148]. Milnor proves this differently using the distance function to a point p rather than L_v.

We then showed that if $f\colon M \to \mathbb{R}$ is a proper Morse function (proper means the inverse image of any compact set is compact) and there are no critical values in $[a, b]$ then $f^{-1}([a, b])$ is diffeomorphic to $f^{-1}(a) \times [0, 1]$, and hence $f^{-1}((-\infty, a])$ is diffeomorphic to $f^{-1}((-\infty, b])$ and $f^{-1}((-\infty, b])$ deformation retracts to $f^{-1}((-\infty, a])$. So despite James's opinion, the only interesting stuff happens at critical points. We started to look at $f^{-1}([c - \epsilon, c + \epsilon])$ where c is a critical value of f and there is only one critical point in $f^{-1}(c)$ of index λ.

11/10 Suppose $f\colon M \to \mathbb{R}$ is a proper Morse function. After a small perturbation we may as well assume each critical value is distinct, i.e., different critical points of f have different critical values. (You can accomplish this by adding ϵu to f where $u = 1$ on a small neighborhood of a critical point and $u = 0$ outside a slightly larger neighborhood. If ϵ is small enough you will add no new critical points but will change the critical value by ϵ.) Now if c is a critical value of f for a critical point of index λ and $[c-\epsilon, c+\epsilon]$ contains no other critical values, we showed that $f^{-1}((-\infty, c+\epsilon])$ is obtained from $f^{-1}((-\infty, c-\epsilon])$ by adding a λ handle, in other words there is an embedding $\nu\colon S^{\lambda-1} \times D^{m-\lambda} \to f^{-1}(c - \epsilon)$ and we obtain $f^{-1}((-\infty, c + \epsilon])$ from $f^{-1}((-\infty, c - \epsilon])$ by gluing $D^\lambda \times D^{m-\lambda}$ with the map ν.

There are a number of consequences of this. A manifold has the homotopy type of a CW complex with one λ cell for each critical point of index λ. (With a little more work one can show that a manifold is a CW complex, at least if it's compact, but for all purposes I know of, homotopy type is good enough.) But more strongly we see that any smooth manifold is a handlebody, i.e., it can be built up by attaching handles as above. This has immense consequences, some of which we will see later. References [**Mi2**, 14-24], [**H**, 156-160]

We can compute the homology of M by taking the homology of the chain complex $\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots$ where $C_i = \mathbb{Z}^{\ell_i}$ and ℓ_i is the number of critical points of index i (if ℓ_i is finite). Next time we will look at the map $\partial_i\colon C_i \to C_{i-1}$ more carefully, it has a nice description in terms of a gradient-like vector field for f.

11/10 We start by looking at handlebodies more carefully. Let S^{n-1} and D^n denote the unit sphere and disc in $\mathbb{R}^n$. If M is an m dimensional manifold with boundary, then to add a k handle to M we choose an embedding $\nu\colon S^{k-1} \times D^{m-k} \to \partial M$. We then take $M' = M \cup_\nu D^k \times D^{m-k}$, we glue $D^k \times D^{m-k}$ to M using ν. To be pedantic we must also smooth out the resulting corner at $\nu(S^{k-1} \times S^{m-k-1})$, but up to diffeomorphism there is only one way to do this. Up to diffeomorphism, the resulting M' only depends on the isotopy class of ν, and in fact only depends on the istopy class of $\nu|_{S^{k-1}\times 0}$ and a trivialization of its normal bundle. We see this by using collaring of ∂M, the isotopy extension theorem, and uniqueness of tubular neighborhoods.

We now look at when we can change the value of a critical point without changing anything else. In particular, if p is a critical point for $f\colon M \to \mathbb{R}$ and v is a gradient-like vector field for f with flow ϕ, then let $D_-(p) = \{x \in M \mid \lim_{t\to\infty} \phi(x,t) = p\}$ and $D_+(p) = \{x \in M \mid \lim_{t\to-\infty} \phi(x,t) = p\}$. If $f(p) = c$ then we know $D_-(p) \subset f^{-1}((-\infty, c])$ and $D_+(p) \subset f^{-1}([c,\infty))$ since $v(f) \geq 0$. Moreover $D_-(p) \cap f^{-1}([c-\epsilon, c])$ is diffeomorphic to a closed disc of dimension λ where λ is the index of p, if ϵ is small enough. We show the following theorem. If $D_-(p) \cap f^{-1}([a,c])$ is diffeomorphic to a closed disc, then there is a Morse function f' so that

a) f' and f have the same critical points and the same indices.
b) f and f' agree ouside an arbitrarily small neighborhood of $D_-(p) \cap f^{-1}([a,c])$.
c) $f'(p) = b$.
d) v is still a gradient-like vector field for f'.

To prove this, let $D = D_-(p) \cap f^{-1}([a,c])$. Take a standard chart $\nu\colon U \to \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$ around p so that in this chart $f\nu^{-1}(x,y) = c - |x|^2 + |y|^2$ and $U \cap D_-(p) = \nu^{-1}(\mathbb{R}^\lambda \times 0)$. Using the flow ϕ we may enlarge this chart to contain D so that $\nu_*(v)$ is still tangent to $\mathbb{R}^\lambda \times 0$. For a very small $\delta > 0$, choose smooth $\alpha\colon \mathbb{R} \to [a-c, 0]$ and $\beta\colon \mathbb{R} \to [0,1]$ so that $\beta(0) = 1$, $\beta(t) = 0$ for $t \geq \delta$, $\beta'(t) \leq 0$, $\alpha(0) = a - c$, $\alpha(t) = 0$ for $t \geq c - a + \delta$, $0 \leq \alpha'(t) < 1$, and α and β are constant in a neighborhood of 0. Define $f'(x) = f(x)$ for $x \notin U$ and $f'\nu^{-1}(x,y) = c - |x|^2 + |y|^2 + \alpha(|x|^2)\beta(|y|^2)$. Then f' has the required properties.

item11/21 I am in the process of writing up what we have been doing. Meanwhile today we talked about the degree of a map which is nicely explained in [**Mi1**, 26-31]. We also talked about how to compute homology of a relative CW complex, see [**Br**, 200-206].

Suggestions for further topics: pseudoholomorphic curves.

Bibliography

[Br] **Bredon**, Topology and Geometry
[BJ] **Bröker & Janich**, Introduction to Differential Topology
[BT] **Bott & Tu**, Differential Forms in Algebraic Topology
[H] **Hirsch**, Differential Topology
[H2] **Hirsch**, Immersions of manifolds, Trans. Amer. Math. Soc. 93 1959 242-276.
[H3] **Hirsch**, On imbedding differentiable manifolds in euclidean space, Ann. of Math. (2) 73 1961 566-571.
[Ki] **Kinsey**, Topology of Surfaces
[Mi1] **Milnor**, Topology from the Differentiable Viewpoint
[Mi2] **Milnor**, Morse Theory
[MS] **Milnor & Stasheff**, Characteristic Classes
[W] **Warner**, Foundations of Differentiable Manifolds and Lie Groups