Math 340

HW Solutions

2.3 14, 24
2.5 2, 20
2.6 3, 18

2.3 #14

$f(x,y) = x^2y + e^{\frac{y}{x}} \qquad a=(1,0)$

$\nabla f(a) = \left(\frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a)\right)$

$\frac{\partial f}{\partial x} = 2xy - \frac{y}{x^2}e^{\frac{y}{x}} \qquad \frac{\partial f}{\partial y} = x^2 + \frac{1}{x}e^{\frac{y}{x}}$

$\frac{\partial f}{\partial x}(a) = 2(1)(0) - \frac{0}{(1)}e^{\frac{0}{1}} = 0 \qquad \frac{\partial f}{\partial y}(a) = 1 + 1\cdot e^{\frac{0}{1}} = 2$

$\nabla f(a) = (0, 2)$

2.3 #24

$f(x,y) = (x^2y, x+y^2, \cos\pi xy)$

$f_1 = x^2y$
$f_2 = x+y^2$
$f_3 = \cos\pi xy$

$a = (2,-1)$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}$$

$$Df = \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ -\pi y\sin\pi xy & -\pi x\sin\pi xy \end{bmatrix}$$

$$Df(a) = \begin{bmatrix} 2(2)(-1) & 2^2 \\ 1 & 2(-1) \\ -\pi(-1)\sin(-2\pi) & -2\pi\sin(-2\pi) \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}$$

2.5 #2

$f(x,y) = \sin(xy) \qquad x = s+t \quad y = s^2+t^2$

(a) find $\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ by substitution

$$f(x,y) = \sin((s+t)(s^2+t^2)) = \sin(s^3+st^2+ts^2+t^3)$$

$$\frac{\partial f}{\partial s} = (\cos(s^3+st^2+ts^2+t^3))(3s^2+t^2+2ts)$$

$$\frac{\partial f}{\partial t} = (\cos(s^3+st^2+ts^2+t^3))(2st+s^2+3t^2)$$

(b) by chain rule.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$= (y\cos(xy))(1) + (x\cos(xy))\,2s$$

$$= (\cos(xy))(y + 2s.x)$$

$$= (\cos(s^3+st^2+ts^2+t^3))(s^2+t^2 + 2s^2+2st)$$

$$= (\cos(s^3+st^2+ts^2+t^3))(t^2+3s^2+2st)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

$$= (y\cos(xy))(1) + (x\cos(xy)\,2t$$

$$= (\cos(s^3+st^2+ts^2+t^3))(s^2+t^2 + 2ts+2t^2)$$

$$= (\cos(s^3+st^2+ts^2+t^3))(s^2+3t^2+2ts)$$

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2.5 #20

$$D(f\circ x)(t_0) = Df(x_0)D_x(t_0)$$

$$t_0 = (1,-1,3) \qquad x = g \qquad x_0 = g(t_0) = (2,5).$$

$$D(f\circ g)(t_0) = Df(2,5)\,D_g(t_0) = Df(2,5)\begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix}$$

$$f(x,y) = (2xy,\ 3x-y+5).$$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix} \qquad Df(x_0) = \begin{bmatrix} 2\cdot 5 & 2\cdot 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}$$

$$D(f\circ g)(t_0) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix}$$

2.6 #3

$$f(x,y) = x^2 - 2x^3y + 2y^3 \qquad a = (2,-1) \qquad \vec{u} = \frac{i+2j}{\sqrt{5}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$\|\vec{u}\| = \sqrt{\frac{1}{5}+\frac{4}{5}} = 1$$

$$\nabla f = (2x - 6x^2y,\ -2x^3 + 6y^2)$$

$$\nabla f(a) = (4 - 6(4)(-1),\ -2(8) + 6(1))$$

$$= (28, -10)$$

$$\text{Directional Derivative } [Df_u(a)] = \nabla f(a)\cdot\frac{\vec{u}}{\|u\|} = \nabla f(a)\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= \left(\frac{28}{\sqrt{5}} + \frac{-20}{\sqrt{5}}\right) = \frac{8}{\sqrt{5}}$$

2.6 #18

$$2xz + yz - x^2y + 10 = 0 \qquad (x_0, y_0, z_0) = (1, -5, 5)$$

$$f = 2xz + yz - x^2y + 10$$

$$\nabla f = (2z - 2xy, z - x^2, 2x + y)$$

$$\nabla f(1, -5, 5) = (10 - 2(1)(-5), 5 - 1, 2 - 5) = (20, 4, -3)$$

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$$

$$(20, 4, -3) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$(20, 4, -3) \cdot (x - 1, y - (-5), z - 5) = 0$$

$$(20, 4, -3) \cdot (x - 1, y + 5, z - 5) = 0$$

$$20x - 20 + 4y + 20 - 3z + 15 = 0$$

$$20x + 4y - 3z + 15 = 0$$