div, grad, and curl as linear transformations

Let X be an open[1] subset of $\mathbb{R}^n$. Let SF_X denote the vector space of real valued functions on X (i.e., scalar fields) and let VF_X denote the vector space of vector fields on X. Colley defines maps[2] grad: $\mathrm{SF}_X \to \mathrm{VF}_X$ and div: $\mathrm{VF}_X \to \mathrm{SF}_X$. If $n = 3$ Colley also defines curl: $\mathrm{VF}_X \to \mathrm{VF}_X$.

The first thing to note is that div, grad, and curl are all linear transformations, since for example $\mathrm{grad}(f+g) = \mathrm{grad} f + \mathrm{grad} g$ and $\mathrm{grad}(cf) = c\,\mathrm{grad} f$. Recall from Colley that divcurl $= 0$ and curlgrad $= 0$. So we will look at a little linear algebra where the composition of linear transformations is 0.

So suppose $T\colon U \to V$ and $S\colon V \to W$ are linear transformations and $ST = 0$. It may help to think of T and S as matrices whose product is 0, although this will not apply to the infinite dimensional examples we are studying here. Let $NS(S) = \{\alpha \in V \mid S\alpha = 0\}$ be the null space of S which is a subspace of V. Let $T(U) = \{\alpha \in V \mid \alpha = T\beta \text{ for some } \beta \in U\}$ be the range of T, which is also a subspace of V. Since $ST = 0$ we know that $T(U) \subset NS(S)$. If S and T were matrices, $NS(S)$ would be the null space of S and $T(U)$ would be the column space of T. So $T(U) \subset NS(S)$ is the same as saying that each column of T is in the null space of S which I hope you can convince yourself of. (Note $0 = Col_i(ST) = SCol_iT$).

If $X \subset Y$ is a subspace then there is a vector space X/Y called the quotient space which I will not describe[3]. If X and Y are both subsets of some $\mathbb{R}^n$, you can think of X/Y as all vectors in X which are perpendicular to Y. It turns out that $\dim(X/Y) = k$ if and only if there are k linearly independent vectors $\beta_1, \beta_2, \ldots, \beta_k$ in X so that any vector α in X can be written uniquely as $\alpha = c_1\beta_1 + c_2\beta_2 + \cdots + c_k\beta_k + \beta$ where β is in Y. For example if $X = \mathbb{R}^3$ and Y is the plane with equation $x + 2y + z = 0$, then $\dim(X/Y) = 1$ since if for example $\beta_1 = (1,0,0)$ then any vector (a,b,c) in $\mathbb{R}^3$ can be written $(a,b,c) = (a+2b+c)\beta_1 + (-2b-c,b,c)$ and $(-b-c,b,c)$ is in Y. Moreover there is no other way to write (a,b,c) as a multiple of β_1 plus a vector in Y.

Now let us go back to our vector fields. We have vector spaces $H^1(X) = NS(\mathrm{curl})/\mathrm{grad}(SF_X)$ and $H^2(X) = NS(\mathrm{div})/\mathrm{curl}(VF_X)$. A consequence of something called DeRham cohomology is that $\dim(H^1(X))$ is the number of 'tunnels' running through X and $\dim(H^2(X))$ is the number of 'holes' in X. Note that $H^1(X) = 0$ means that $NS(\mathrm{curl}) = \mathrm{grad}(SF_X)$ which means that a vector field on X is conservative if and only if its curl is 0. So if $X \subset \mathbb{R}^3$ has no tunnels then a vector field on X is conservative if and only if its curl is 0.

We say that X is star shaped if there is a point $\alpha \in X$ so that for any $\beta \in X$ the entire line segment from α to β is in X. I will show that if X is star shaped and G is a vector field on X so $\mathrm{curl}G = 0$, then $G = \mathrm{grad}h$ for some h. In fact here is a formula for h, $h(x) = \int_0^1 G(\alpha + t(x-\alpha)) \cdot (x-\alpha)\,dt =$ the work integral for G on the line segment from α to x. By the product rule and the ability to differentiate under the integral sign,

$$\partial h/\partial x_i = \int_0^1 \partial(G(\alpha + t(x-\alpha)) \cdot (x-\alpha))/\partial x_i\,dt$$

$$= \int_0^1 \partial(G(\alpha + t(x-\alpha)))/\partial x_i \cdot (x-\alpha) + G(\alpha + t(x-\alpha)) \cdot \mathbf{e}_i\,dt$$

$$= \int_0^1 t\partial G/\partial x_i(\alpha + t(x-\alpha))) \cdot (x-\alpha) + G_i(\alpha + t(x-\alpha))\,dt$$

[1] With lots of hassle, one could do all this if X is not open, but it is not worth obscuring the main ideas to do so. By the way, I suggest you ignore the footnotes on your first reading. The main text is meant to give the essential ideas involved, sometimes telling a white lie, but this is all corrected in the footnotes.

[2] This is not quite true. Since the maps involve derivatives, you need differentiability of the functions and vector fields. Here are some options on how to correct this:

a) restrict div, grad, and curl to the subspace of differentiable functions and vector fields, or
b) define SF_X^k and VF_X^k to be the subspaces of k times differentiable functions and vector fields and then, for example, define div: $\mathrm{VF}_X^1 \to \mathrm{SF}_X^0$ and note that it takes VF_X^k to SF_X^{k-1} for all k, or
c) define SF_X and VF_X to be the vector spaces of infinitely differentiable scalar fields and vector fields.

[3] Okay, so I will describe it. Consider the equivalence relation on X where we say $x \sim x'$ if $x - x' \in Y$. The vectors in X/Y are equivalence classes under this equivalence relation. If $x \in X$, let $[x]$ denote its equivalence class in X/Y. We define $[x] + [x'] = [x + x']$ and $c[x] = [cx]$.

But since curl$G = 0$ we know $\partial G_i/\partial x_j = \partial G_j/\partial x_i$ so grad$G_i = \partial G/\partial x_i$. So combining all this we get

$$\partial h/\partial x_i = \int_0^1 t\partial G/\partial x_i(\alpha + t(x-\alpha))) \cdot (x-\alpha) + G_i(\alpha + t(x-\alpha))\, dt$$

$$= \int_0^1 t\text{grad}G_i \cdot (x-\alpha) + G_i(\alpha + t(x-\alpha))\, dt$$

$$= \int_0^1 dtG_i(\alpha + t(x-\alpha))/dt\, dt = tG_i(\alpha + t(x-\alpha))\Big]_0^1 = G_i(x)$$

So in the end, $G = \text{grad}h$.

Note that we did not use dimension 3 at all above, except to say curl$G = 0$ implies $\partial G_i/\partial x_j = \partial G_j/\partial x_i$. So in fact it proves in all dimensions that if G is a vector field with star shaped domain, then G is conservative if and only if $\partial G_i/\partial x_j = \partial G_j/\partial x_i$ for all i and j.

Let us do an example where this does not hold. Suppose X is the region inside the sphere $x^2+y^2+z^2 = 9$ and outside the cylinder $x^2+y^2 = 1$. So X looks like a cored apple and it has one tunnel running through it, the (empty) core of the apple. Consider the vector field $F(x,y,z) = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)$. Then

$$\text{curl}F(x,y,z) = (\partial(x(x^2+y^2)^{-1})/\partial x - \partial(-y(x^2+y^2)^{-1})/\partial y)\mathbf{k}$$

$$= ((x^2+y^2)^{-1} - x(x^2+y^2)^{-2}(2x) + (x^2+y^2)^{-1} - y(x^2+y^2)^{-2}(2y))\mathbf{k} = 0$$

On the other hand, if $F = \text{grad}g$ then $\partial g/\partial x = -y(x^2+y^2)^{-1}$, $\partial g/\partial y = x(x^2+y^2)^{-1}$, and $\partial g/\partial z = 0$. The last equation says that g is a function of x and y alone and does not depend on z. Integrating the first equation we see that $g(x,y) = \tan^{-1}(y/x) + C(y)$ for some function C. Plugging into the second equation we get $C(y)$ is a constant. So in polar coordinates, $g(x,y) = \theta$. The problem is that this is not continuous, as you go around X, g goes from 0 to 2π and then must immediately jump back from 2π to 0. So F is not conservative, there is no differentiable function g so that $F = \text{grad}g$.

On the other hand, I claim that if $G(x,y,z)$ is any vector field on X with curl$G = 0$ then there is a c and a $g(x,y,z)$ so that $G = cF + \text{grad}g$, so $H^1(X)$ has dimension 1. This may be easiest to see using the cylindrical coordinate version given on page 219 of the text. Note $F = (-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j})/r^2 = \mathbf{e}_\theta/r$. If $G = G_r\mathbf{e}_r + G_\theta\mathbf{e}_\theta + G_z\mathbf{e}_z$ then curl$G = 0$ means $\partial G_z/\partial\theta = \partial rG_\theta/\partial z$, $\partial G_z/\partial r = \partial G_r/\partial z$, and $\partial G_r/\partial\theta = \partial rG_\theta/\partial r$. We want to find c and $g(r,\theta,z)$ so that $G = cF + \text{grad}g$ which means $G_r = \partial g/\partial r$, $G_\theta = c/r + (1/r)\partial g/\partial\theta$, and $G_z = \partial g/\partial z$. Letting $\bar{G} = rG_\theta$ our equations become $\partial G_z/\partial\theta = \partial\bar{G}/\partial z$, $\partial G_z/\partial r = \partial G_r/\partial z$, $\partial G_r/\partial\theta = \partial\bar{G}/\partial r$, $G_r = \partial g/\partial r$, $\bar{G} = c + \partial g/\partial\theta$, and $G_z = \partial g/\partial z$. In other words, the vector field $(G_r, \bar{G}, G_z)$ in the r,θ,z space has 0 curl. It is defined on the region $0 \le \theta \le 2\pi$, $1 \le r \le 3$, $-\sqrt{9-r^2} \le Z \le \sqrt{9-r^2}$ which is star shaped and hence we may find a function $h(r,\theta,z)$ so that $\partial h/\partial r = G_r$, $\partial h/\partial\theta = \bar{G}$, and $\partial h/\partial z = G_z$.

Consider $f(r,z) = h(r,2\pi,z) - h(r,0,z)$. Then

$$\partial f(r,z)/\partial r = \partial h(r,2\pi,z)/\partial r - \partial h(r,0,z)/\partial r = G_r(r,2\pi,z) - G_r(r,0,z) = 0$$

Similarly, $\partial f(r,z)/\partial z = 0$. So $f(r,z)$ does not change when either r or z changes, so $f(r,z)$ is some constant d. Let $g(r,\theta,z) = h(r,\theta,z) - \frac{d}{2\pi}\theta$. Then $g(r,2\pi,z) = g(r,0,z)$ for all r and z so g is a well defined continuous function on X. Moreover grad$g = G - \frac{d}{2\pi}F$ which is what we wanted to show.