BLOWING UP DIFFERENTIAL EQUATIONS IN THE PLANE

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ABSTRACT. An abstract

1. INTRODUCTION

Most beginning differential equations courses include a section on qualitative behavior of ordinary differential equations in the plane. The basic idea is that near a stationary point, the qualitative behavior of the solution curves is often the same as that of the linearized system, which in turn can be determined from the eigenvalues of the linearized system. The student is presented with a table such as that below summarizing the possibilities. (I omit comments on stability).

Roots of characteristic polynomial	Linear type	Almost linear type
$r_1 > r_2 > 0$	Improper Node	Improper Node
$r_1 < r_2 < 0$	Improper Node	Improper Node
$r_1 > 0 > r_2$	Saddle Point	Saddle Point
$r_1 = r_2 \neq 0$	Proper or Improper Node	Proper or Improper Node or Spiral Point
$r_1, r_2 = a \pm bi$, $a, b \neq 0$	Spiral Point	Spiral Point
$r_1, r_2 = \pm bi$, $b \neq 0$	Center	Center or Spiral Point

Note the ambiguity in the case of equal roots. This is rather trying for the student learning this material. When for amusement I tried to prove these results I was suprised to find that if the differential equations are smooth, this ambiguity disappears. In particular, the whole complicated table can be replaced by the statement that as long as no eigenvalue is zero or pure imaginary the solution curves will behave qualitatively like those of the linearized system. Besides being a more precise result, how much easier this is for the student to learn! (Moreover the ambiguity in the behavior of linear systems with one eigenvalue is also easily resolved – if the matrix is diagonal there is a proper node, otherwise it is improper).

I should make it clear from the outset what I mean by the vague statement that the solution curves behave qualitatively like those of the linearized system. In the proper node case, I mean that for each ray from the stationary point, there is exacly one solution curve with that limiting tangent ray. In the improper node equal eigenvalue case, I mean that there are only two possible limiting tangent rays (opposite in direction) and moreover there is a smooth curve through the stationary point (made up of two solution curves and the stationary point) so that solution curves on one side have one limiting tangent ray, and those on the other side have

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the other limiting tangent ray. In other words, they look like the solutions to the linearized system except that the straight line solution curves could be curvy, but tangencies at the origin are preserved.

Note that I do not claim that there is a smooth change of coordinates taking one to the other. Technically, there is a continuous change of coordinates which is differentiable away from the origin and which preserves tangencies at the origin.

2. Blowing up

The proof I give below uses the algebraic notion of blowing up, so I will start with an explanation of what this is. First a vague definition – to blow up a point p in $\mathbb{R}^n$, replace the point p with the projective space of lines through p. This projective space which replaces p is called the exceptional divisor. There is a map from the blowup to $\mathbb{R}^n$ which is the identity off the exceptional divisor and crushes the divisor to p. Blowing up is of great use in the field of algebraic geometry. You can think of blowing up as veiwing with a variable power microscope whose power goes to infinity as we approach p. Thus it takes infinitesimal information at p and makes it visible.

In $\mathbb{R}^2$ you can think of blowing up the origin as taking polar coordinates (without the restriction $r \geq 0$) and identifying (r, θ) with $(-r, \theta + \pi)$. After the identification of (r, θ) with $(-r, \theta + \pi)$, all points except the origin have just one set of polar coordinates but the origin has a whole circle's worth, parameterized by the angle θ. We can identify this circle with the space of lines through the origin by identifying a line with the angle it makes with the positive x axis. So if we blow up a point in $\mathbb{R}^2$, we get a Möbius band whose central circle is the exceptional divisor.

We will not use polar coordinates here but instead will think of the blowup of the origin in $\mathbb{R}^2$ as being a manifold covered by two coordinate charts. In one chart we have coordinates x, u and the blowup map is $(x, u) \mapsto (x, ux)$. In this chart, the exceptional divisor is $x = 0$ and the point $(0, u)$ corresponds to the line with slope u. In the second chart we have coordinates v, y and the map is $(v, y) \mapsto (vy, y)$. The exceptional divisor is $y = 0$ and the point $(v, 0)$ corresponds to the line with slope $1/v$.

Nice curves lift to the blowup as the following lemma shows.

Lemma 2.1. *Suppose $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ is a smooth curve, $\alpha^{-1}(\mathbf{0}) = 0$, and $\alpha'(0) \neq \mathbf{0}$. If $\pi\colon B \to \mathbb{R}^2$ is the blowup of the origin $\mathbf{0}$, then α lifts to a curve $\beta\colon \mathbb{R} \to B$ so that $\pi \circ \beta = \alpha$. Moreover, $\beta(0)$ must be in the exceptional divisor and it corresponds to the line in the direction of $\alpha'(0)$, i.e., the line tangent to the curve α at $\mathbf{0}$.*

Proof. Let α_1 and α_2 be the coordinates of α. Since $\alpha_i(0) = 0$ there are smooth $\gamma_i\colon \mathbb{R} \to \mathbb{R}$ so that $\alpha_i(t) = t\gamma_i(t)$ and $\alpha'(0) = (\gamma_1(0), \gamma_2(0))$. If $\gamma_1(0) \neq 0$ then α lifts to $t \mapsto (\alpha_1(t), \gamma_2(t)/\gamma_1(t))$ in the x, u chart. If $\gamma_2(0) \neq 0$ then α lifts to $t \mapsto (\gamma_1(t)/\gamma_2(t), \alpha_2(t))$ in the v, y chart. □

The requirement that $\alpha'(0) \neq \mathbf{0}$ is not necessary to ensure lifting. For example any analytic curve will lift to an analytic curve in the blowup.

3. Ordinary Differential Equations in the Plane

Now we are ready to talk about differential equations in the plane. Consider a differential equation $\mathbf{z}' = H(\mathbf{z})$ where $\mathbf{z} = (x, y)$ is a point in the plane, and H is smooth. For simplicity, take smooth to mean C^∞ but you can get by with much less. Suppose that $H(\mathbf{0}) = \mathbf{0}$ so we have a stationary point at the origin. By Taylor's theorem we may write $H(\mathbf{z}) = A\mathbf{z} + K(\mathbf{z})$ for some matrix A so that K vanishes with order greater than one, i.e., $\partial K/\partial x$ and $\partial K/\partial y$ both vanish at $\mathbf{0}$. We suppose that $\det A \neq 0$, otherwise we would expect K to have a qualitative effect on the behavior of the system.

Since $\det(A) \neq 0$ we know that $\mathbf{0}$ is an isolated stationary point, but for convenience we suppose that $\mathbf{0}$ is the only stationary point, i.e., $H(\mathbf{0}) = \mathbf{0}$ and $H(\mathbf{z}) \neq \mathbf{0}$ if $\mathbf{z} \neq \mathbf{0}$. We also assume for convenience that at least one eigenvalue of A has nonpositive real part. (If not, we may negate H and turn all arrows backwards.)

Our technique will be to lift the differential equation to a differential equation on the blowup B of $\mathbb{R}^2$ at the origin. Solution curves of the lifted differential equation will be lifts of solutions of the original. Analysis of the solution curves in B leads to information on the solution curves in $\mathbb{R}^2$.

The first step is to do an appropriate linear change of coordinates $\mathbf{w} = S\mathbf{z}$ for a nonsingular matrix S. Then $\mathbf{w}' = S\mathbf{z}' = SAS^{-1}\mathbf{w} + SK(S^{-1}\mathbf{w})$. So by an appropriate choice of S we may suppose that A is in real Jordan canonical form, i.e., the system is of the form:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = J \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$$

where F and G have order 2 or more at the origin, and J is a matrix in real Jordan canonical form. (By real Jordan canonical form I mean that J is in Jordan canonical form $J = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right]$ or $J = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$ if the eigenvalues of A are real, and $J = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ if the eigenvalues of A are $a \pm b\sqrt{-1}$.)

Since F has order 2 or more we know for example that the function $(x, u) \mapsto F(x, ux)/x^2$, which appears not to be defined when $x = 0$ has a continuous extension to the whole x, u plane, see section 6 below. We will often use such extensions below without comment.

4. The case of only one eigenvalue

We will immediately focus on the case where J has only one eigenvalue, so either $J = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]$ or $J = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$. At the end of the paper we'll look briefly at the other cases which are also made accessable using blowups, but in those cases the results are well known and I find the proofs I give less interesting.

First let us do the case where $J = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]$. After blowing up we have two coordinate charts, the first has coordinates x, u where $y = ux$ and the second has coordinates v, y where $x = vy$.

Note that

$$\begin{aligned} x' &= \lambda x + F(x, ux) \\ u' &= y'/x - yx'/x^2 \\ &= G(x, ux)/x - uF(x, ux)/x \end{aligned} \tag{4.1}$$

At first glance we appear to be worse of than before, since now the differential equation (4.1) has stationary points on the whole line $x = 0$. But since F and G are order 2 or more, both $F(x, ux)$ and $G(x, ux)$ are divisible by x^2. Consequently, both x' and u' are divisible by x. So we may divide the right sides of the differential equations (4.1) by x and obtain differential equations which have the same slope

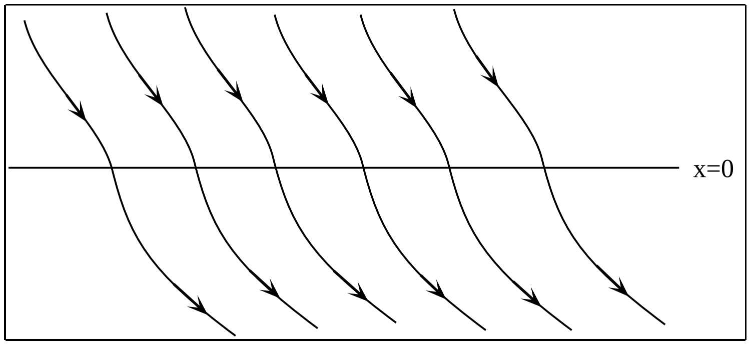

FIGURE 1. Solution curves to (4.2)

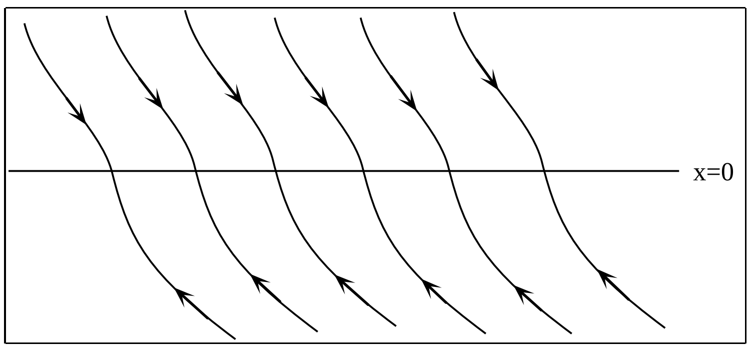

FIGURE 2. Solution curves to (4.1)

field as (4.1). So their solutions are just reparameterizations of those of (4.1). After dividing by x, the differential equations (4.1) become:

$$\begin{aligned} x' &= \lambda + F(x, ux)/x \\ u' &= G(x, ux)/x^2 - uF(x, ux)/x^2 \end{aligned} \tag{4.2}$$

Note that $x' = \lambda \neq 0$ on the whole line $x = 0$ so there are no stationary points of (4.2) on the line $x = 0$ and moreover, the solution curves all cross the line $x = 0$ transversely, see Figure 1. The solution curves of (4.1) are just reparameterizations of the solution curves of (4.2), where the velocity is multiplied by x. In particular, each solution curve of (4.2) is divided into three solution curves of (4.1): one with $x > 0$, one with $x < 0$, and the third a stationary point with $x = 0$. The direction of the $x < 0$ is reversed since the velocity is multiplied by a negative quantity. So the solutions to (4.1) would look like Figure 2. In particular, each point on the line $x = 0$ is the limit of exactly two nonstationary solution curves of (4.1). Although we just did one chart, by symmetry the other chart has the same behavior. So downstairs in $\mathbb{R}^2$ this means that for each tangent line to the origin, there are exactly two solution curves with that tangent line. As a bonus, the union of those two curves and the origin is a smooth curve. Thus the solution curves look like those of Figure 3.

(exercise) Determine the relation between the curvatures of the solution curves at the origin and the slopes the solution curves of (4.2) at the line $x = 0$, which are

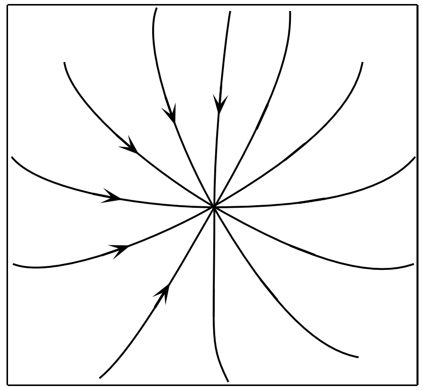

FIGURE 3. Solution curves in $\mathbb{R}^2$

in turn determined by the second derivatives of F and G at the origin. Show that the slope at $(0, u)$ is given by a polynomial of degree at most three in u and thus either all solution curves have zero curvature or else three or fewer pairs of solution curves have zero curvature.

Now let us do the case where $J = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$. After blowing up we have two coordinate charts, the first has coordinates x, u where $y = ux$ and the second has coordinates v, y where $x = vy$.

Let us first look at the second chart where $x = vy$. Then

$$\begin{aligned} v' &= x'/y - xy'/y^2 \\ &= 1 + F(vy, y)/y - vG(vy, y)/y \\ y' &= \lambda y + G(vy, y) \end{aligned} \tag{4.3}$$

Since $v' = 1$ and $y' = 0$ on the line $y = 0$ we see that there are no stationary points in this chart and moreover the line $y = 0$ is a solution curve. Additionally, $y'/y \approx \lambda < 0$ near $y = 0$, so the solution curves off of, but near, $y = 0$ get closer and closer to $y = 0$.

So the only interesting stuff will happen in the first chart where $y = ux$. Note that in this chart

$$\begin{aligned} x' &= \lambda x + ux + F(x, ux) \\ u' &= y'/x - yx'/x^2 \\ &= G(x, ux)/x - u^2 - uF(x, ux)/x \end{aligned} \tag{4.4}$$

If we restrict to the line $x = 0$ we see that $x' = 0$ and $u' = -u^2$ so the behavior is certainly not linear at the stationary point $(0, 0)$.

Take a small rectangle $R = \{(x, u) \colon |x| \le \delta, |u| \le \epsilon\}$ around the point $(0, 0)$. In particular, take any ϵ in $(0, |\lambda|)$ and then choose a small δ. If δ is chosen small enough, we see that trajectories enter the rectangle all along the edges $u = \epsilon$ and $x = \pm\delta$, and they leave the rectangle all along the edge $u = -\epsilon$. At this point we need to rule out the possibility that some trajectory entering the edge $u = \epsilon$ will exit at the edge $u = -\epsilon$, no matter how small we take δ. This would mean the trajectories in $\mathbb{R}^2$ behaved qualitatively like a spiral point or possibly something strange (if only trajectories on one side of the line $x = 0$ did this).

To rule out this possibility, we blow up a second time. Let $B_{xu} \subset B$ denote the x, u chart of B. Let $\rho\colon C \to B_{xu}$ be the blowup of B_{xu} at the point $(0,0)$. So C has a chart with coordinates x, s and $u = sx$ and a chart with coordinates t, u and $x = tu$. In the first chart, our differential equations become:

$$
\begin{aligned}
x' &= \lambda x + sx^2 + F(x, sx^2) \\
s' &= y'/x^2 - 2yx'/x^3 \\
&= -\lambda s + G(x, sx^2)/x^2 - 2s^2x - 2sF(x, sx^2)/x
\end{aligned}
\tag{4.5}
$$

Note that if we evaluate $G(x, sx^2)/x^2$ on $x = 0$ we just get $\partial^2 G/\partial x^2(0,0)/2$. So on the new exceptional divisor $x = 0$, the differential equation restricts to $x' = 0$, $s' = -\lambda s + a$ where $a = \partial^2 G/\partial x^2(0,0)/2$ so we get a single stationary point at $x = 0$, $s = a/\lambda$. Setting $z = s - a/\lambda$ the linearized differential equation at this stationary point is just

$$
\begin{aligned}
x' &= \lambda x \\
z' &= bx - \lambda z
\end{aligned}
\tag{4.6}
$$

for some constant b of no importance[1]. In particular, the stationary point is a saddle. This means that there is a pair of solution curves α_1, α_2 which limit on the stationary point at $x = 0, s = a/\lambda$. Then $\rho \circ \alpha_i$ are solution curves in B which limit on the point $(0,0)$ in the x, u chart and moreover have a limiting tangent line with slope λ/a.

So if we choose δ small enough, the solution curves $\rho \circ \alpha_i$ will enter the rectangle R along the top and bottom edges $x = \pm\delta$. In fact they will enter close to the points $x = \pm\delta$, $u = \pm a\delta/\lambda$. In particular, it will be impossible for any solution curve to enter the rectangle R at the edge $u = \epsilon$ and leave at $u = -\epsilon$ since it would have to cross one of those two special solutions $\rho \circ \alpha_i$.

So the upshot is this, let $x = \delta$, $u = c_+$ and $x = -\delta$, $u = c_-$ be the points at which these two trajectories enter the rectangle R. (We have $c_\pm \sim \pm a\delta/\lambda$.) Then we may divide the boundary of R into two pieces, $R_+ = \delta \times [c_+, \epsilon] \cup -\delta \times [c_-, \epsilon] \cup [-\delta, \delta] \times \epsilon$ and $R_- = \delta \times [-\epsilon, c_+) \cup -\delta \times [-\epsilon, c_-) \cup [-\delta, \delta] \times -\epsilon$. Then trajectories which enter R on R_+ will approach $(0,0)$ as a limit and those which enter on R_- will exit on $[-\delta, \delta] \times -\epsilon$. However, note that these trajectories entering at R_- are merely delaying their limit to $(0,0)$. Looking at them in the v, y chart, they travel steadily nearer the exceptional divisor. Eventually, after going once around the exceptional divisor, they must reenter R on the edge $u = \epsilon$, closer to $x = 0$ than they exited (and on the opposite side of $x = 0$), and then will limit to the point $(0,0)$. If we look at what this means down in the unblownup $\mathbb{R}^2$, the trajectories entering at R_- spiral halfway around the origin before finally limiting on the origin tangent to the x axis. On the other hand, those entering at R_+ do not spiral, but immediately limit on the origin tangent to the x axis.

(exercise) The special trajectories $\pi \circ \rho \circ \alpha_i$ which are the dividing line between these two behaviors are, to second order, close to the parabola ???.

(exercise) I did not justify that the special trajectories $\pi \circ \rho \circ \alpha_i$ are in fact the dividing line between these two behaviors. Conceivably, some trajectory in B_{xu} could enter on R_- and limit to $(0,0)$ without exiting R. Show that this does not

[1] I believe $b = \partial^3 G/\partial x^3(0,0)/6 + (a/\lambda)\partial^2 G/\partial x \partial y(0,0) - 2(a/\lambda)^2 - (a/\lambda)\partial^2 F/\partial x^2(0,0)$.

happen by looking more carefully at the blowup C. You will have to look at the t, u chart.

5. The well known cases with unequal eigenvalues

Next let us look at the case where $J = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right]$ with $\lambda < \mu < 0$. Blow up the origin. Let us first look in the chart where $y = ux$. Note that in this chart

$$\begin{aligned} x' &= \lambda x + F(x, ux) \\ u' &= y'/x - yx'/x^2 \\ &= (\mu - \lambda)u + G(x, ux)/x - uF(x, ux)/x \end{aligned} \tag{5.1}$$

The only stationary point in this chart is at $x = u = 0$. Since $\mu - \lambda > 0 > \lambda$ this is a saddle point.

Let us now look at the second chart where $x = vy$. Then

$$\begin{aligned} v' &= x'/y - xy'/y^2 \\ &= (\lambda - \mu)v + F(vy, y)/y - vG(vy, y)/y \\ y' &= \mu y + G(vy, y) \end{aligned} \tag{5.2}$$

The only stationary point is at $y = v = 0$. The eigenvalues are $\lambda - \mu$ and μ which are both negative. At first glance we seem to have made no progress since, unless $\lambda = 2\mu$, we are back to the same case we started with. But it turns out all we really care about here is stability, the fact that all trajectories approach the stationary point. Since $(v^2+y^2)' \sim (\lambda-\mu)v^2+\mu y^2 < 0$ for (v, y) close to $(0, 0)$ solution curves which get close enough to $(0, 0)$ will limit on $(0, 0)$. Likewise solution curves which approach closely enough to the exceptional divisor will get closer and closer.

The net result of all this is that there are exactly two solution curves β_1, β_2 in the x, u chart which approach the saddle point $(0, 0)$. The rest of the solution curves close enough to the exceptional divisor eventually get close enough to the the point $(0, 0)$ in the v, y chart that they are forced to limit on it. So down in $\mathbb{R}^2$ we have two solution curves $\pi \circ \beta_1$ with horizontal limiting tangent and the rest of the solution curves which approach the origin have vertical limiting tangent. This is qualitatively like the linear case.

Now let us look at the saddle point case where $J = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right]$ with $\lambda < 0 < \mu$. Blow up the origin. Let us first look in the chart where $y = ux$. As above, we have

$$\begin{aligned} x' &= \lambda x + F(x, ux) \\ u' &= (\mu - \lambda)u + G(x, ux)/x - uF(x, ux)/x \end{aligned} \tag{5.3}$$

The only stationary point in this chart is at $x = u = 0$. Since $\mu - \lambda > 0 > \lambda$ the linearization is is a saddle point.

At first glance this seems to be no improvement, but in fact this saddle point is special enough to admit easy analysis of its behavior. If $\delta > 0$ is chosen small enough, then $x'/x < 0$ in all the rectangle $R = \{(x, u) \colon |x| \le \delta, |u| \le 1\}$, so the trajectories are approaching the exceptional divisor $x = 0$. The trajectories are pointed into the sides $\pm\delta \times [-1, 1]$ of R and pointed out of the other two sides.

For simplicity, let us just look at trajectories in the region $x > 0$. Since trajectories cannot cross each other, we know there are imbeddings $f_+ \colon (0, \delta] \to [-1, 1]$ and $f_- \colon (0, \delta] \to [-1, 1]$ so that the trajectory entering R at $(\delta, f_+(t)$ will exit at $(t, 1)$ and the trajectory entering R at $(\delta, f_-(t)$ will exit at $(t, -1)$. If $c_+ = \lim_{t\to 0} f_+(t)$ and $c_- = \lim_{t\to 0} f_-(t)$ then the trajectories entering R on the whole interval

$\delta \times [c_-, c_+]$ must then approach the origin since their x coordinate decreases and they have nowhere else to go.

So to show we have a saddle point, we must show $c_+ = c_-$. To show this, we will show that $\partial/\partial u$ of the slope u'/x' of the trajectory is negative (if $x > 0$). Consequently, the trajectories diverge from each other in the u direction and we could not have two different trajectories both approaching the origin in the region $x > 0$. Note that $u'/x' = (\mu/\lambda - 1)u/x + h(x,u)$ for some smooth h. So $\partial/\partial u(u'/x') = (\mu/\lambda - 1)/x + \partial h/\partial u(x,u) \sim (\mu/\lambda - 1)/x < 0$ as required.

While the case where the eigenvalues of J are complex could be handled using a blowup, it turns out to be easier to use polar coordinates. So we suppose that $J = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ Then:

$$
\begin{aligned}
r' &= (xx' + yy')/r \\
&= (ax^2 - bxy + xF(x,y) + bxy + ay^2 + yG(x,y))/r \\
&= ar + \cos\theta F(x,y) + \sin\theta G(x,y) \\
\theta' &= (xy' - yx')/r^2 \\
&= (bx^2 + axy + xG(x,y) - axy - by^2 - yF(x,y))/r^2 \\
&= b + \cos\theta G(x,y)/r - \sin\theta F(x,y)/r
\end{aligned}
\tag{5.4}
$$

Since F and G have order 2, we then have $r'/r = a + rD(r,\theta)$ and $\theta' = b + rE(r,\theta)$ for some smooth E and D. We may as well suppose that $b > 0$. Then for r small enough, $r'/r < 0$ and $\theta' > 0$. So for small enough $\delta > 0$ there is a decreasing map $f\colon [0,\delta] \to [0,\delta]$ so that the solution curve through $r = t, \theta = 0$ will pass through the point $r = f(t)$, $\theta = 2\pi$. Since we always have $f(t) < t$ for $t > 0$ we get spiralling behavior of the solution curves.

6. Dividing functions

We have made much use above of such facts as that $f(x,ux)/x^2$ extends to a smooth function at $x = 0$ if f is smooth and order 2 or more. Let us see why this is true. If $f(x,y)$ is smooth, then

$$df(tx,ty)/dt = xf_x(tx,ty) + yf_y(tx,ty)$$

Hence,

$$f(x,y) - f(0,0) = \int_0^1 df(tx,ty)/dt\,dt = xf_1(x,y) + yf_2(x,y)$$

where $f_1(x,y) = \int_0^1 f_x(tx,ty)dt$ and $f_2(x,y) = \int_0^1 f_y(tx,ty)dt$. Repeating this construction for f_1 and f_2 we get

$$f(x,y) = f(0,0) + xf_1(0,0) + yf_2(0,0) + x^2 f_3(x,y) + xyf_4(x,y) + y^2 f_5(x,y)$$

for some smooth f_i. This is just a form of Taylor's theorem with remainder. Now, for example, if f has order 2 or more we have:

$$f(x,ux)/x^2 = f_3(x,ux) + uf_4(x,ux) + u^2 f_5(x,ux)$$

so $f(x,ux)/x^2$ extends to a smooth function at $x = 0$.

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