Square roots and n-th roots of linear operators

Suppose $T: V \to V$ is a linear operator on a complex vector space V. When is there a linear operator $S: V \to V$ so that $S^2 = T$? This shows how the Jordan form gives the answer.

First consider the case $V = \mathbb{C}^n$ and T is multiplication by a Jordan block matrix $J_{n,c}$ with $c \neq 0$. Let $N = J_{n0}/c$ then $T = c(I+N)$ and N is nilpotent, $N^n = 0$. recall the Taylor series for $\sqrt{1+x}$

$$\sqrt{1+x} = 1 + 1/2x + (1/2)(-1/2)/2!x^2 + (1/2)(-1/2)(-3/2)/3!x^3 + \cdots$$

So we know $1 + x = (1 + 1/2x + (1/2)(-1/2)/2!x^2 + (1/2)(-1/2)(-3/2)/3!x^3 + \cdots)^2$. Plug in N for x and we get:

$$I + N = (I + 1/2N + (1/2)(-1/2)/2!N^2 + (1/2)(-1/2)(-3/2)/3!N^3 + \cdots)^2$$

we don't have to worry about convergence, since $N^n = 0$ so the power series has at most $n+1$ terms. So if $A = \sqrt{c}(I + 1/2N + (1/2)(-1/2)/2!N^2 + (1/2)(-1/2)(-3/2)/3!N^3 + \cdots)$, then $A^2 = N$ and we have found a square root of this Jordan block.

Let's do a concrete example where $c = 4$ and $n = 3$. Then $N = \begin{bmatrix} 0 & 0 & 0 \\ 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \end{bmatrix}$, $N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/16 & 0 & 0 \end{bmatrix}$, and $N^3 = 0$. We have $A = 2(I + 1/2N - 1/8N^2) = \begin{bmatrix} 2 & 0 & 0 \\ 1/4 & 2 & 0 \\ -1/64 & 1/4 & 2 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{bmatrix}$.

So if we put T in Jordan form, we can take the square root of all its Jordan blocks, as long as their diagonal entries are nonzero. So for example, if T is nonsingular then 0 is not a characteristic value of T so all its Jordan blocks have a square root, so T has a square root.

So the only thing left is to understand the Jordan blocks when $c = 0$ where the above method will not work. So we assume T is nilpotent and suppose there is an operator S so that $S^2 = T$. Then S is also nilpotent, so we may write V as a direct sum of cyclic subspaces

$$V = Z(\alpha_1; S) \oplus \cdots \oplus Z(\alpha_k; S)$$

The trick to notice is that $Z(\alpha; S) = Z(\alpha; S^2) \oplus Z(S\alpha; S^2) = Z(\alpha; T) \oplus Z(S\alpha; T)$. So we get a cyclic decomposition

$$V = Z(\alpha_1; T) \oplus Z(S\alpha_1; T) \oplus Z(\alpha_2; T) \oplus \cdots \oplus Z(S\alpha_k; T)$$

This cyclic decomposition gives a Jordan form for T which we know is unique. However, note that $\dim Z(S\alpha_i; T)$ either equals $\dim Z(\alpha_i; T)$ or equals $\dim Z(\alpha_i; T) - 1$. So the end result is that if a nilpotent matrix has a square root then you are able to pair up the Jordan blocks so that their size differs by at most 1. (This does not mean there must be an even number of Jordan blocks, since you can pair up a 1×1 block with a 0×0 block which is invisible of course.) The converse is also true. Suppose we can write $V = Z(\alpha_1; T) \oplus \cdots \oplus Z(\alpha_k; T)$ where $\dim Z(\alpha_i; T) = d_i$ and $d_{2i} = d_{2i-1}$ or $d_{2i} = d_{2i-1} - 1$ for each i (and $d_k = 1$ if k is odd). Then define S by $ST^j\alpha_{2i-1} = T^j\alpha_{2i}$ and $ST^j\alpha_{2i} = T^{j+1}\alpha_{2i-1}$ (and $S\alpha_k = 0$ if k is odd). Then $S^2 = T$.

To look at a concrete example, suppose the B is a matrix in Jordan form with Jordan blocks $J_{3,2}$, $J_{2,1}$, $J_{3,0}$, $J_{2,0}$, and $J_{1,0}$. Then $A = \sqrt{B}$ exists since we may find the square roots of the first two blocks directly (since their diagonals are nonzero), and we may pair up $J_{3,0}$ with $J_{2,0}$, and since $J_{1,0}$ is 1×1. We set $A\varepsilon_6 = \varepsilon_9$, $A\varepsilon_7 = \varepsilon_{10}$, $A\varepsilon_8 = 0$, $A\varepsilon_9 = \varepsilon_7$, $A\varepsilon_{10} = \varepsilon_8$, $A\varepsilon_{11} = 0$. The Jordan form of A has Jordan blocks $J_{3,\sqrt{2}}$, $J_{2,1}$, $J_{5,0}$, and $J_{1,0}$ (although A itself is not in Jordan form).

There are similar results for n-th roots of matrices. Instead of pairing up the Jordan blocks for the nilpotent part, you need to divide them in groups of n whose sizes do not differ by more than one, and so if k is not a multiple of n the leftover group with less than n is only 1×1 blocks. So the above example B does not have any cube roots or indeed any n-th roots for $n > 2$.