Math 241 Exam 1 Sample 5 Solutions

1. Given the following data:

$$P = (1, 2, 3)$$
$$Q = (4, 10, 2)$$
$$\bar{a} = 1\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k}$$
$$\bar{b} = -3\,\hat{\imath} + 2\,\hat{\jmath} + 1\,\hat{k}$$

(a) Find a vector perpendicular to both $\overrightarrow{PQ}$ and $\bar{a}$. [10 pts]

Solution: We have $\overrightarrow{PQ} = 3\,\hat{\imath} + 8\,\hat{\jmath} - 1\,\hat{k}$ so a vector perpendicular to both would be $\overrightarrow{PQ} \times \bar{a} = -14\,\hat{\imath} + 5\,\hat{\jmath} - 2\,\hat{k}$.

(b) Find the projection of $\bar{b}$ onto $\bar{a}$. [5 pts]

Solution: We have

$$Pr_{\bar{a}}\bar{b} = \frac{\bar{a} \cdot \bar{b}}{\bar{a} \cdot \bar{a}}\bar{a} = \frac{-2}{9}(1\,\hat{\imath} + 2\,\hat{\jmath} - 2\,\hat{k})$$

(c) Find the unit vector in the direction of $\overrightarrow{PQ}$. [5 pts]

Solution: The answer is

$$\frac{\overrightarrow{PQ}}{||\overrightarrow{PQ}||} = \frac{3\,\hat{\imath} + 8\,\hat{\jmath} - 1\,\hat{k}}{\sqrt{3^2 + 8^2 + (-1)^2}} = \frac{3\,\hat{\imath} + 8\,\hat{\jmath} - 1\,\hat{k}}{\sqrt{74}}$$

2. (a) Find the distance between the point $(3, 2, 1)$ and the plane $2x - 3y + 10z = 20$. Simplify. [12 pts]

Solution: The normal vector for the plane is $\bar{N} = 2\,\hat{\imath} - 3\,\hat{\jmath} + 10\,\hat{k}$ and a point on the plane is $P = (10, 0, 0)$. We have $Q = (3, 2, 1)$ off the plane. The distance is then

$$
\begin{aligned}
\text{dist} &= \frac{|\overrightarrow{PQ} \cdot \bar{N}|}{||\bar{N}||} \\
&= \frac{|(-7\,\hat{\imath} + 2\,\hat{\jmath} + 1\,\hat{k}) \cdot (2\,\hat{\imath} - 3\,\hat{\jmath} + 10\,\hat{k})|}{\sqrt{2^2 + (-3)^2 + 10^2}} \\
&= \frac{|-14 - 6 + 10|}{\sqrt{113}} \\
&= \frac{10}{\sqrt{113}}
\end{aligned}
$$

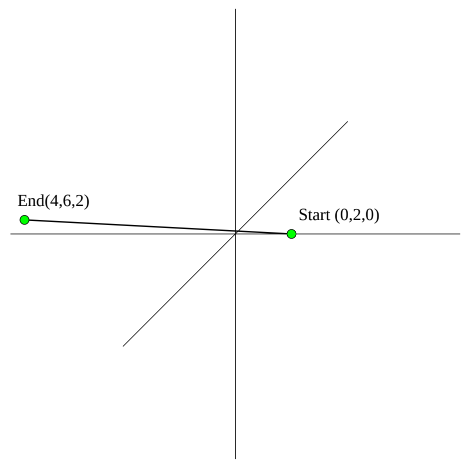
(b) Find the symmetric equation for the line through the points $(2, -1, 4)$ and $(0, 1, 4)$. [8 pts]

Solution: The line (one version) has $x = 2 - 2t$, $y = -1 + 2t$ and $z = 4$. If we solve for t in the first two and set equal we get

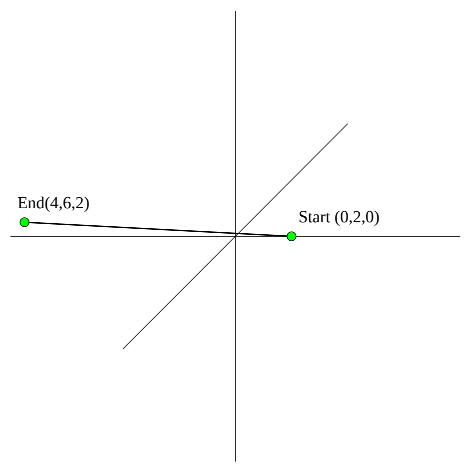
$$\frac{x-2}{-2} = \frac{y+1}{2} \quad , \quad z = 4$$

3. (a) Find the point where the line through $(0, 2, 1)$ and $(3, 4, 5)$ passes through the plane $z = 0$. [6 pts]
Solution: The line has parametric equation $x = 3t$, $y = 2 + 2t$ and $z = 1 + 4t$. This hits the plane when $1 + 4t = 0$ or $t = -\frac{1}{4}$ hence at $x = 3(-1/4)$, $y = 2 + 2(-1/4)$ and $z = 0$.

(b) Sketch the VVF $\bar{r}(t) = 2t\,\hat{\imath} + (2 - 4t)\,\hat{\jmath} + t\,\hat{k}$ for $0 \leq t \leq 2$. Indicate direction. [6 pts]
Solution: This is a straight line from $\bar{r}(0) = 0\,\hat{\imath} + 2\,\hat{\jmath} + 0\,\hat{k}$ to $\bar{r}(2) = 4\,\hat{\imath} - 6\,\hat{\jmath} + 2\,\hat{k}$. More or less like this:



(c) Give a parametrization of the oriented semi-ellipse shown here. [8 pts]



Solution: We have $\bar{r}(t) = -2\cos t\,\hat{\imath} + 4\sin t\,\hat{\jmath}$ with $0 \leq t \leq \pi$.

4. (a) Assuming a and b are positive constants calculate the curvature of the ellipse $\bar{r}(t) = a\cos t\,\hat{\imath} + b\sin t\,\hat{\jmath}$ at $t = \frac{\pi}{2}$. [10 pts]

Solution:

We have $\bar{v} = -a\sin t\,\hat{\imath} + b\cos t\,\hat{\jmath}$ and $\bar{a} = -a\cos t - b\sin t$. Then $\bar{v} \times \bar{a} = ab\,\hat{k}$ and so the curvature is $\kappa = \frac{||\bar{v}\times\bar{a}||}{||\bar{v}||^3} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$ hence at $t = \frac{\pi}{2}$ we have $\kappa = \frac{ab}{a^3} = \frac{b}{a^2}$

(b) Calculate the length of the curve $\bar{r}(t) = 2t\,\hat{\imath} + t^2\,\hat{\jmath} + \ln t\,\hat{k}$ for $1 \leq t \leq 2$. Simplify. [10 pts]

Solution: We have $\bar{r}'(t) = 2\,\hat{\imath} + 2t\,\hat{\jmath} + \frac{1}{t}\,\hat{k}$ and so

$$\begin{aligned}
\text{Length} &= \int_1^2 ||\bar{r}'(t)||\;dt \\
&= \int_1^2 \sqrt{4 + 4t^2 + \frac{1}{t^2}}\;dt \\
&= \int_1^2 \sqrt{\left(2t + \frac{1}{t}\right)^2}\;dt \\
&= \int_1^2 2t + \frac{1}{t}\;dt \\
&= t^2 + \ln|t|\Big|_1^2 \\
&= (4 + \ln 4) - (1 + \ln 1) \\
&= 3 + \ln 4
\end{aligned}$$

5. (a) Find the position vector satisfying $\bar{a}(t) = 2\,\hat{\imath} + 2\,\hat{\jmath}$, $\bar{v}(0) = 1\,\hat{\imath} - 2\,\hat{\jmath}$ and $\bar{r}(1) = 3\,\hat{\imath} + 5\,\hat{\jmath}$. [13 pts]

Solution: First:

$$\begin{aligned} \bar{a}(t) &= 2\,\hat{\imath} + 2\,\hat{\jmath} \\ \bar{v}(t) &= \int 2\,\hat{\imath} + 2\,\hat{\jmath}\;dt \\ \bar{v}(t) &= 2t\,\hat{\imath} + 2t\,\hat{\jmath} + \bar{C} \end{aligned}$$

Then $\bar{v}(0) = 0\,\hat{\imath} + 0\,\hat{\jmath} + \bar{C} = 1\,\hat{\imath} - 2\,\hat{\jmath}$ so $\bar{C} = 1\,\hat{\imath} - 2\,\hat{\jmath}$ and so $\bar{v}(t) = (2t+1)\,\hat{\imath} + (2t-2)\,\hat{\jmath}$. Then

$$\begin{aligned} \bar{v}(t) &= (2t+1)\,\hat{\imath} + (2t-2)\,\hat{\jmath} \\ \bar{r}(t) &= \int (2t+1)\,\hat{\imath} + (2t-2)\,\hat{\jmath}\;dt \\ \bar{r}(t) &= (t^2+t)\,\hat{\imath} + (t^2-2t)\,\hat{\jmath} + \bar{D} \end{aligned}$$

Then $\bar{r}(1) = 2\,\hat{\imath} - 1\,\hat{\jmath} + \bar{D} = 3\,\hat{\imath} + 5\,\hat{\jmath}$ so $\bar{D} = 1\,\hat{\imath} + 6\,\hat{\jmath}$ and so $\bar{r}(t) = (t^2+t+1)\,\hat{\imath} + (t^2-2t+6)\,\hat{k}$.

(b) Sketch the plane $x + 2y + 3z = 12$. Label the three intercepts with their coordinates. [7 pts]

Solution: We have: