Math 241 Exam 2 Sample 4 Solutions

1. Define $f(x, y) = x^2 + 6xy - 2y^3$.

   (a) We use $\bar{u} = \frac{1}{\sqrt{2}}\,\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath}$ and we have $f_x(x, y) = 2x + 6y$ and $f_y(x, y) = 6x - 6y^2$ and so

   $$D_{\bar{u}}(2, 2) = \frac{1}{\sqrt{2}}(2(2) + 6(2)) - \frac{1}{\sqrt{2}}(6(2) - 6(2)^2)$$

   (b) We have

   $$\begin{aligned} 2x + 6y &= 0 \\ 6x - 6y^2 &= 0 \end{aligned}$$

   The first gives $x = -3y$ which we plug into the second to get $-18y - 6y^2 = 0$ or $-6y(3+y) = 0$ which gives $y = -3$ or $y = 0$.

   If $y = -3$ we have $x = -3(-3) = 9$ yielding $(9, -3)$.

   If $y = 0$ we have $x = -3(0) = 0$ yielding $(0, 0)$.

   (c) We have $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -12y$ and $f_{xy}(x, y) = 6$ so that

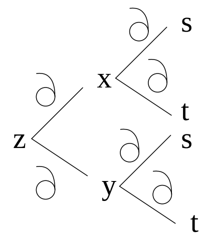
   $$D(x, y) = (2)(-12y) - (6)^2$$

   Then:

   For $(9, -3)$ we have $D(9, -3) = (2)(36) - 36 = +$ so $f_{xx}(9, -3) = +$ and it's a relative min.

   For $(0, 0)$ we have $D(0, 0) = (2)(0) - 36$ and it's a saddle point.

2. (a) Our function tree is:



and so

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= (2xy + 1)(1) + (x^2)(\sin s)$$
$$= 2(2s + t)(t \sin s) + 1 + (2s + t)^2 \sin s$$

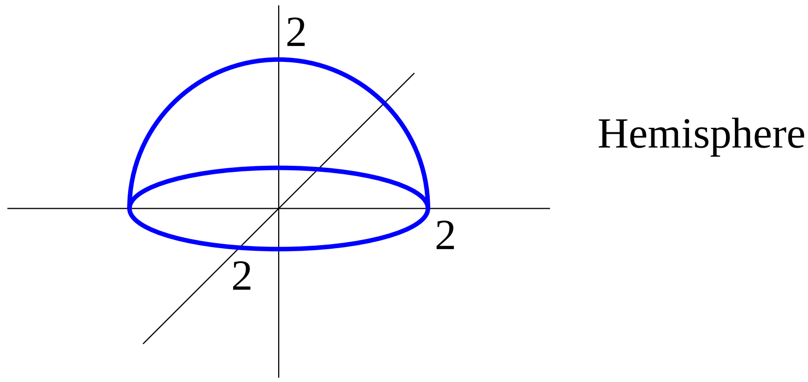
(b) The surface is the level surface for $z = x^2y + y^2$ or $f(x, y, z) = x^2y + y^2 - z$ so we find

$$\nabla f(x, y, z) = 2xy\,\hat{\imath} + (x^2 + 2y)\,\hat{\jmath} - \,\hat{k}$$
$$\nabla f(1, 2, 6) = 4\,\hat{\imath} + 5\,\hat{\jmath} - \,\hat{k}$$

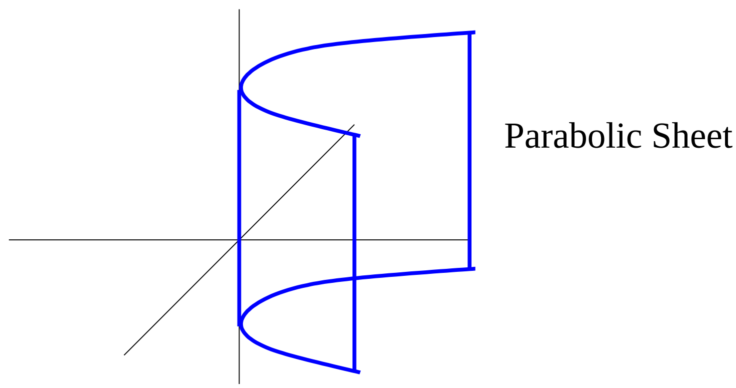
So $\bar{N} = 4\,\hat{\imath} + 5\,\hat{\jmath} - \,\hat{k}$ and using the point $(1, 2, 6)$ we have

$$4(x - 1) + 5(y - 2) - 1(z - 6) = 0$$

3. (a) The figure is:



(b) The figure is:



(c) The equation is $(x-2)^2+y^2+z^2=4$.

(d) The equation is $z=4-x^2-y^2$.

4. We first find the critical points and set equal to zero:

$$f_x(x,y) = y + 2x = 0$$
$$f_y(x,y) = x = 0$$

This yields the single point $(0,0)$ and $f(0,0) = 0$.

On the bottom edge $y = 0$ so $f = x^2$.
The minimum is 0 at $(0,0)$ and the maximum is 4 at $(2,0)$.

On the right edge $x = 2$ so $f = 2y + 4$.
The minimum is 4 at $(2,0)$ and the maximum is 8 at $(2,2)$.

On the diagonal edge $y = x$ so $f = 2x^2$.
The minimum is 0 at $(0,0)$ and the maximum is 8 at $(2,2)$.

Overall the minimum is 0 and the maximum is 8.

5. We have $f(x,y) = xy + 2y$ and $g(x,y) = x^2 + y^2$. Our three equations are then:

$$\begin{aligned} y &= \lambda(2x) \\ x + 2 &= \lambda(2y) \\ x^2 + y^2 &= 4 \end{aligned}$$

Call these (A), (B) and (C). Then from (A) we have $x = 0$ or $\lambda = \frac{y}{2x}$. We can't have $x = 0$ because (A) would give $y = 0$ and together these contradict (C).

So $\lambda = \frac{y}{2x}$ and then plugging into (B) yields

$$\begin{aligned} x + 2 &= \frac{y}{2x}(2y) \\ x + 2 &= \frac{y^2}{x} \\ x^2 + 2x &= y^2 \end{aligned}$$

Put this into (C) to get

$$\begin{aligned} x^2 + x^2 + 2x &= 4 \\ x^2 + x - 2 &= 0 \\ (x-1)(x+2) &= 0 \end{aligned}$$

which gives us $x = 1$ or $x = -2$.

If $x = -2$ then (C) gives us $y = 0$ for the point $(-2, 0)$
If $x = 1$ then (C) gives us points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$.

Then check these points:
$f(-2, 0) = 0$
$f(1, \sqrt{3}) = 3\sqrt{3}$ This is the max!
$f(1, -\sqrt{3}) = -3\sqrt{3}$ This is the min!