

Math 241 Exam 4 Sample 3 Solutions

1. We're looking for

$$\iint_{\Sigma} (0 \hat{i} + x \hat{j} + z \hat{k}) \cdot \bar{n} \, dS$$

We parametrize Σ by

$$\bar{r}(x, y) = x \hat{i} + y \hat{j} + (9 - x^2) \hat{k} \quad \text{with} \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 2$$

and then

$$\begin{aligned}\bar{r}_x &= 1 \hat{i} + 0 \hat{j} - 2x \hat{k} \\ \bar{r}_y &= 0 \hat{i} + 1 \hat{j} + 0 \hat{k} \\ \bar{r}_x \times \bar{r}_y &= 2x \hat{i} + 0 \hat{j} + 1 \hat{k}\end{aligned}$$

Note that these vectors have a positive \hat{k} component which matches the orientation of Σ . So we have

$$\begin{aligned}\iint_{\Sigma} (0 \hat{i} + x \hat{j} + z \hat{k}) \cdot \bar{n} \, dS &= + \iint_R (0 \hat{i} + x \hat{j} + (9 - x^2) \hat{k}) \cdot (2x \hat{i} + 0 \hat{j} + 1 \hat{k}) \, dA \\ &= \iint_R 9 - x^2 \, dA \\ &= \int_0^2 \int_0^2 9 - x^2 \, dy \, dx \\ &= \int_0^2 9y - x^2 y \Big|_0^2 \, dx \\ &= \int_0^2 18 - 2x^2 \, dx \\ &= 18x - \frac{2}{3}x^3 \Big|_0^2 \\ &= 18(2) - \frac{2}{3}(2)^3\end{aligned}$$

2. (a) Since \bar{F} is conservative with potential function $f(x, y) = \frac{1}{2}x^2y^2 + x$ and so

$$\begin{aligned}\int_C x^2y + 1 \, dx + xy^2 \, dy &= f(3, 3) - f(1, -2) \\ &= \left[\frac{1}{2}(3)^2(3)^2 + 3 \right] - \left[\frac{1}{2}(1)^2(-2)^2 + 1 \right]\end{aligned}$$

- (b) We parametrize the line segment as

$$\bar{r}(t) = 5t \hat{i} + 4t \hat{j} \quad \text{with} \quad 0 \leq t \leq 1$$

and then

$$\begin{aligned}\bar{r}'(t) &= 5 \hat{i} + 4 \hat{j} \\ ||\bar{r}'(t)|| &= \sqrt{41}\end{aligned}$$

and then

$$\int_C 2x + y \, ds = \int_0^1 [2(5t) + 4t] \sqrt{41} \, dt$$

3. By Green's Theorem we have

$$\int_C 2x \, dx + x^2 \, dy = \iint_R 2x - 0 \, dA$$

where R is the region inside the curve. This region is parametrized best in polar coordinates so we have

$$\begin{aligned} \int_C 2x \, dx + x^2 \, dy &= \iint_R 2x - 0 \, dA \\ &= \int_0^{\pi/2} \int_1^2 2r \cos \theta \, r \, dr d\theta \\ &= \int_0^{\pi/2} \left. \frac{2}{3} r^3 \cos \theta \right|_1^2 d\theta \\ &= \int_0^{\pi/2} \frac{14}{3} \cos \theta \, d\theta \\ &= \left. \frac{14}{3} \sin \theta \right|_0^{\pi/2} \\ &= \frac{14}{3} [\sin(\pi/2) - \sin(0)] \\ &= \frac{14}{3} \end{aligned}$$

4. The best Σ would be the portion of the plane $x + y = 5$ inside the cylinder. The orientation of Σ would be toward the right.

Since $\vec{F}(x, y, z) = x \hat{i} + 3 \hat{j} + 2y \hat{k}$ we have $\nabla \times \vec{F} = 2 \hat{i} + 0 \hat{j} + 0 \hat{k}$.

Then Stokes's Theorem tells us

$$\int_C x \, dx + 3 \, dy + 2y \, dz = \iint_{\Sigma} (2 \hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot \vec{n} \, dS$$

Since Σ is inside the cylinder it's a good choice to parametrize it as

$$\vec{r}(r, \theta) = r \cos \theta \hat{i} + (5 - r \cos \theta) \hat{j} + r \sin \theta \hat{k} \quad \text{with} \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

Then

$$\begin{aligned} \vec{r}_r &= \cos \theta \hat{i} - \cos \theta \hat{j} + \sin \theta \hat{k} \\ \vec{r}_\theta &= -r \sin \theta \hat{i} + r \sin \theta \hat{j} + r \cos \theta \hat{k} \\ \vec{r}_r \times \vec{r}_\theta &= -r \hat{i} - r \hat{j} + 0 \hat{k} \end{aligned}$$

Note that these vectors point have negative \hat{k} component and hence point left, opposite to that for Σ . So we have

$$\begin{aligned} \iint_{\Sigma} (2 \hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot \vec{n} \, dS &= - \iint_R (2 \hat{i} + 0 \hat{j} + 0 \hat{k}) \cdot (-r \hat{i} - r \hat{j} + 0 \hat{k}) \, dA \\ &= - \iint_R -2r \, dA \\ &= - \int_0^{2\pi} \int_0^2 -2r \, dr \, d\theta \end{aligned}$$

5. If D is the solid cube then the Divergence Theorem gives us

$$\begin{aligned}\iint_{\Sigma} (5x \hat{i} + 2y \hat{j} - 2z \hat{k}) \cdot \bar{n} \, dS &= \iiint_D (5 + 2 - 2) \, dV \\ &= 5 \iiint_D 1 \, dV \\ &= 5 \text{ Volume of Cube} \\ &= 5(8) \\ &= 40\end{aligned}$$