Math 241 Exam 4 Sample 4 Solutions

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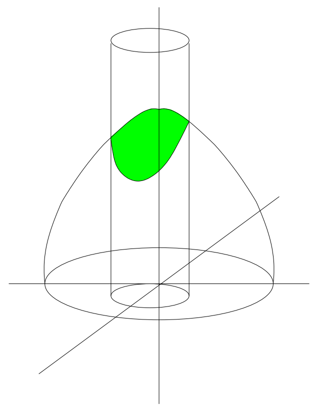
1. Let Σ be the portion of $z = 16 - x^2 - y^2$ inside the cylinder $r = 2\cos\theta$ and with upwards orientation. Draw a picture of Σ and find the rate at which the fluid $\bar{F}(x, y, z) = 0\,\hat{\imath} + x\,\hat{\jmath} + 0\,\hat{k}$ is flowing through Σ. [20 pts]

   Stop when you have an iterated double integral.

   Solution:



We parametrize Σ as $\bar{r}(r, \theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + (16 - r^2)\,\hat{k}$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $0 \le r \le 2\cos\theta$. Then

$$\begin{aligned}
\bar{r}_r &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} - 2r\,\hat{k} \\
\bar{r}_\theta &= -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\,\hat{k} \\
\bar{r}_r \times \bar{r}_\theta &= 2r^2\cos\theta\,\hat{\imath} + 2r^2\sin\theta\,\hat{\jmath} + r\,\hat{k}
\end{aligned}$$

Note that these vectors have positive $\hat{k}$-component so they match the orientation for Σ. Then we have:

$$\begin{aligned}
\iint_\Sigma (0\,\hat{\imath} + x\,\hat{\jmath} + 0\,\hat{k}) \cdot \bar{n}\ dS &= +\iint_R (0\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\,\hat{k}) \cdot (2r^2\cos\theta\,\hat{\imath} + 2r^2\sin\theta\,\hat{\jmath} + r\,\hat{k})\ dA \\
&= \iint_R 2r^3 \sin\theta\cos\theta\ dA \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} 2r^3\sin\theta\cos\theta\ dr\ d\theta
\end{aligned}$$

2. (a) Evaluate $\int_C y\ dx + (x+1)\ dy$ where C is parametrized by $\bar{r}(t) = e^t \sin(\pi t)\,\hat{\imath} + e^t \cos(\pi t)\,\hat{\jmath}$ for $0 \leq t \leq \frac{1}{2}$. [7 pts]

Stop when you have an unsimplified numerical answer.

Solution:

The vector field is conservative with potential function $f(x,y) = xy + y$. The start point is $\bar{r}(0) = 0\,\hat{\imath} + 1\,\hat{\jmath}$ or $(0,1)$ and the end point is $\bar{r}(1/2) = e^{1/2}\,\hat{\imath} + 0\,\hat{\jmath}$ or $(\sqrt{e}, 0)$. Then by the FToLI we have

$$\int_C y\ dx + (x+1)\ dy = f(\sqrt{e}, 0) - f(0,1) = 0 - 1 = -1$$

(b) Find the mass of the wire C, where C is the line segment in the xy-plane joining $(2,0)$ to $(5,4)$ and the density is $f(x,y) = 3xy$. [13 pts]

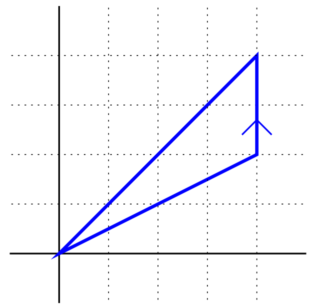
Stop when you have an unsimplified numerical answer.

Solution:

The curve C is parametrized by $\bar{r}(t) = (2+3t)\,\hat{\imath} + (0+4t)\,\hat{\jmath}$ for $0 \leq t \leq 1$. Then $\bar{r}'(t) = 3\,\hat{\imath} + 4\,\hat{\jmath}$ so $||\bar{r}'(t)|| = \sqrt{25} = 5$ and so the mass is

$$\begin{aligned}
\int_C f(x,y)\ ds &= \int_C 3xy\ ds \\
&= \int_0^1 3(2+3t)(0+4t)5\ dt \\
&= \int_0^1 120 + 180t\ dt \\
&= 120t + 90t^2 \Big|_0^1 \\
&= 120 + 90
\end{aligned}$$

3. Evaluate $\int\limits_C x^2 \ dx + 3xy \ dy$ where C is the curve shown in the picture. [20 pts]



Solution:

By Green's Theorem we can change to an integral over R which is the region inside C. We parametrize R as vertically simple. Therefore:

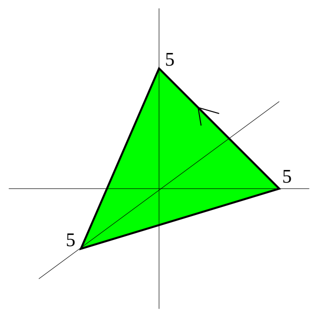
$$\begin{aligned}
\int_C x^2 \ dx + 3xy \ dy &= \iint_R 3y - 0 \ dA \\
&= \iint_R 3y \ dA \\
&= \int_0^4 \int_{\frac{1}{2}x}^{x} 3y \ dy \ dx \\
&= \int_0^4 \frac{3}{2}y^2 \Big|_{\frac{1}{2}x}^{x} \ dx \\
&= \int_0^4 \frac{3}{2}x^2 - \frac{3}{8}x^2 \ dx \\
&= \int_0^4 \frac{9}{8}x^2 \ dx \\
&= \frac{3}{8}x^3 \Big|_0^4 \\
&= \frac{3}{8}(4)^3
\end{aligned}$$

4. Let C be the triangle with vertices $(5, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 5)$ oriented clockwise when viewed from above. Use Stokes' Theorem to find the work done on a particle by the force $\bar{F}(x, y, z) = yz\,\hat{\imath} + y\,\hat{\jmath} + xy\,\hat{k}$ as the particle traverses the curve C. Include a picture of C and Σ (these can be together on one picture). [25 pts]

Stop when you have an iterated double integral.

Solution:

The triangle is the boundary of the portion of the plane $x + y + z = 5$ in the first octant so this is Σ. The counterclockwise orientation of C induces an upwards orientation on Σ.



The surface Σ is parametrized by $\bar{r}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath} + (5 - x - y)\,\hat{k}$ with $0 \le x \le 5$ and $0 \le y \le 5 - x$. This gives us

$$
\begin{aligned}
\bar{r}_x &= 1\,\hat{\imath} + 0\,\hat{\jmath} - 1\,\hat{k} \\
\bar{r}_y &= 0\,\hat{\imath} + 1\,\hat{\jmath} - 1\,\hat{k} \\
\bar{r}_x \times \bar{r}_y &= 1\,\hat{\imath} + 1\,\hat{\jmath} + 1\,\hat{k}
\end{aligned}
$$

which matches the orientation of Σ.

Then we have $\nabla \times \bar{F} = x\,\hat{\imath} + 0\,\hat{\jmath} - z\,\hat{k}$ and so all together:

$$
\begin{aligned}
\int_C (yz\,\hat{\imath} + y\,\hat{\jmath} + xy\,\hat{k}) \cdot d\bar{r} &= \iint_\Sigma (x\,\hat{\imath} + 0\,\hat{\jmath} - z\,\hat{k}) \cdot \bar{n}\ dS \\
&= + \iint_R (x\,\hat{\imath} + 0\,\hat{\jmath} - (5 - x - y)\,\hat{k}) \cdot (1\,\hat{\imath} + 1\,\hat{\jmath} + 1\,\hat{k})\ dA \\
&= \iint_R x - (5 - x - y)\ dA \\
&= \int_0^5 \int_0^{5-x} 2x + y - 5\ dy\ dx
\end{aligned}
$$

5. Let Σ be the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the sphere $x^2 + y^2 + z^2 = 9$ as well as the portion of the sphere inside the cone. [15 pts]
Find the rate at which the fluid $\bar{F}(x, y, z) = y\,\hat{\imath} + x\,\hat{\jmath} + z^2\,\hat{k}$ is flowing inwards through Σ.
Stop when you have an iterated triple integral.

Solution:

By the Divergence Theorem and considering the orientation of Σ we have:

$$\begin{aligned}
\iint_{\Sigma} (y\,\hat{\imath} + x\,\hat{\jmath} + z^2\,\hat{k}) \cdot \bar{n}\ dS &= -\iiint_{D} 0 + 0 + 2z\ dV \\
&= -\int_0^{2\pi} \int_0^{\pi/4} \int_0^{3} 2(\rho\cos\phi)\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta
\end{aligned}$$