Math 241 Fall 2010 Final Exam Solutions *THIS IS A ROUGH DRAFT; IF YOU FIND AN ERROR, TELL ME SO THAT I CAN FIX IT FOR YOUR CLASSMATES!* aross@math.umd.edu

---

1. There are lots of ways that we could complete this problem, but the fastest is a 3-step process:

   i) Find a normal vector, $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

   ii) Find a base point (x_0, y_0, z_0)

   iii) Plug into the standard "point-slope"-style formula $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

   To find $\mathbf{n}$, we'll cross two vectors in the plane. We have lots of choices, but two example vectors in the plane are $(1-3)\mathbf{i} + (1-(-1))\mathbf{j} + (0-(-2))\mathbf{k}$ which points from the first point to the second point, and $(0-3)\mathbf{i} + (1-(-1))\mathbf{j} + (2-(-2))\mathbf{k}$ which points from the first point to the third point. Simplifying and crossing, we get

   $$(-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \times (-3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

   Taking the second point as our base point (arbitrarily), we have

   $$\boxed{4(x-1) + 2(y-1) + 2(z-0) = 0}$$

   which in "slope-intercept"-style looks like $\boxed{2x + y + z = 3}$ (either form is fine, or any equation which is equivalent to these).

---

2. (a) If we start with the unit circle centered at the origin and stretch it out to be twice as wide and thrice as tall, we have the given ellipse. Thus a quick-and-dirty parametrization in stretched polar coordinates is

   $$\boxed{\mathbf{r}(t) = 2\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}}$$

   defined on the interval $0 \le t < 2\pi$, for example. This is by no means a unique solution.

   (b) We have

   $$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j} \quad \text{and}$$
   $$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{v}'(t) = -2\cos(t)\mathbf{i} - 3\sin(t)\mathbf{j}$$

   so that

   $$\kappa(t) = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{\|(6\sin^2(t) + 6\cos^2(t))\mathbf{k}\|}{\sqrt{(4\sin^2(t) + 9\cos^2(t)}^3} = \boxed{\frac{6}{(4 + 5\cos^2(t))^{3/2}}}.$$

   (c) The fraction above is at its maximum when $\cos^2(t)$ is as small as possible, that is when $\cos^2(t) = 0$ i.e. $t = \frac{(2n+1)\pi}{2}$ for integers n. Conversely, the fraction is at its minimum when the denominator is as big as possible, that is when $\cos^2(t) = 1$, i.e. $t = n\pi$ for integers n. These maximal and minimal values are

   $$\kappa\left(\frac{\pi}{2}\right) = \frac{6}{4^{3/2}} = \boxed{\frac{3}{4}} \quad \text{and} \quad \kappa(0) = \frac{6}{9^{3/2}} = \boxed{\frac{2}{9}}$$

   respectively. To determine where they are attained, we look at the position function $\mathbf{r}$. The maximum occurs at $\mathbf{r}(\pi/2)$ (the top intercept, $\boxed{(0,3)}$) and at $\mathbf{r}(3\pi/2)$ (the bottom intercept, $\boxed{(0,-3)}$). The minimum occurs at $\mathbf{r}(0)$ (the right intercept, $\boxed{(2,0)}$) and at $\mathbf{r}(\pi)$ (the left intercept, $\boxed{(-2,0)}$). This makes sense; since the ellipse is taller than it is wide, it has to curve (bend; turn; change direction) more sharply at its top and bottom than on its sides.

3. Since we can't evaluate this integral directly, let's change the order of integration (we'll go vertically simple). Our region is a right triangle. Its bottom edge is the line $y = 0$, and its top edge is the line $y = x$. The range of x values is $[0, 1]$. Hence we can compute

$$\int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 x e^{x^2} dx = \int_0^1 \frac{1}{2} e^u du = \frac{1}{2}(e^1 - e^0) = \boxed{\frac{e-1}{2}}$$

where we have made the substitution $u = x^2$ with $du = 2xdx$.

---

4. Recall that the surface area of a surface Σ is exactly $\iint_\Sigma 1 dS$. If Σ is the portion of the sphere $x^2 + y^2 + z^2 = 16$ on which $x^2 + y^2 \leq 1$, then we can either parameterize Σ in cylindrical or spherical coordinates quite easily. The fact that we have a sphere segment suggests spherical coordinates, but our constraint suggests cylindrical; in general, the constrain will win out, so we'll use cylindrical here. First though, note that Σ has two separate components (call the northern polar cap Σ_N and the southern one Σ_S) which are mirror images; in particular they have the same surface area. Thus to get the total surface area of Σ, we can just double the surface area of one or the other (we'll work with Σ_N).

   Now let's look at our cylindrical coordinates. Our equation becomes $r^2 + z^2 = 16$ so that we have $z = \pm\sqrt{16 - r^2}$ (since we've picked the top, $z = +\sqrt{16 - r^2}$). So Σ_N is parameterized by

$$\mathbf{r}(r, \theta) = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j} + \sqrt{16 - r^2}\,\mathbf{k}$$

   on the domain R where $0 \leq r \leq 1$ and $0 \leq \theta < 2\pi$. The first partials of $\mathbf{r}$ are

$$\mathbf{r}_r(r, \theta) = \cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j} + \frac{-r}{\sqrt{16 - r^2}}\mathbf{k}$$

$$\mathbf{r}_\theta(r, \theta) = -r\sin(\theta)\mathbf{i} + r\cos(\theta)\mathbf{j} + 0\mathbf{k}$$

   with cross product

$$(\mathbf{r}_r \times \mathbf{r}_\theta)(r, \theta) = \frac{r^2\cos(\theta)}{\sqrt{16 - r^2}}\mathbf{i} + \frac{r^2\sin(\theta)}{\sqrt{16 - r^2}}\mathbf{j} + (r\cos^2(\theta) + r\sin^2(\theta))\mathbf{k}.$$

   This cross product has magnitude

$$\begin{aligned}
\|\mathbf{r}_r \times \mathbf{r}_\theta\| &= \sqrt{\frac{r^4\cos^2(\theta)}{16 - r^2} + \frac{r^4\sin^2(\theta)}{16 - r^2} + r^2} \\
&= \sqrt{\frac{r^4}{16 - r^2} + \frac{16r^2 - r^4}{16 - r^2}} \\
&= \sqrt{\frac{16r^2}{16 - r^2}} \\
&= 4r(16 - r^2)^{-1/2}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\iint\limits_{\Sigma} 1 dS &= 2 \iint\limits_{\Sigma_N} 1 dS \\
&= 2 \iint\limits_{R} 1 \|\mathbf{r}_r \times \mathbf{r}_\theta\| dA \\
&= 2 \int_0^{2\pi} \int_0^{1} 4r(16 - r^2)^{-1/2}\, dr d\theta \\
&= 2 \int_0^{2\pi} \int_{16}^{15} -2u^{-1/2}\, du d\theta \\
&= 2 \int_0^{2\pi} 4(\sqrt{16} - \sqrt{15}) d\theta \\
&= \boxed{16\pi(4 - \sqrt{15})}
\end{aligned}$$

where $u = 16 - r^2$ so that $du = -2r dr$.

---

5. We can use the divergence theorem to convert this flux integral to a triple integral over the interior solid D. We have

$$\begin{aligned}
\iiint\limits_{D} \nabla \cdot \mathbf{F}\, dV &= \iiint\limits_{D} (y - 1 + 1)\, dV \\
&= \int_0^{1} \int_0^{1-y^2} \int_0^{1+x} y\, dz dx dy \\
&= \int_0^{1} \int_0^{1-y^2} y(1 + x)\, dx dy \\
&= \int_0^{1} y \left(1 - y^2 + \frac{(1 - y^2)^2}{2}\right) dy \\
&= \frac{1}{2} \int_0^{1} (3y - 4y^3 + y^5)\, dy \\
&= \frac{1}{2} \left(\frac{3}{2} - \frac{4}{4} + \frac{1}{6}\right) \\
&= \boxed{\frac{1}{3}}.
\end{aligned}$$

6. (a) First, let's check that both surfaces pass through the point $(0, 2, 2)$. We have

$$2 = 4(0)^2 + (2)^2 - 2$$

and

$$2 = (0)^2 + 4(2) - 6$$

so that checks out. Now recall that the gradient of a function is normal to its level sets. Let's take a function like

$$g_1(x, y, z) = 4x^2 + y^2 - z$$

which has our first surface as one of its level sets (although it is not the only such function). Its gradient is

$$\nabla g_1(x, y, z) = 8x\mathbf{i} + 2y\mathbf{j} - 1\mathbf{k}$$

and so the vector

$$\nabla g_1(0, 2, 2) = 0\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$$

is normal to the first surface at the desired point. Thus it is also normal to the tangent plane there.

Similarly for the second surface, take

$$g_2(x, y, z) = x^2 + 4y - z$$

so that

$$\nabla g_2(x, y, z) = 2x\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$$

and so the vector

$$\nabla g_2(0, 2, 2) = 0\mathbf{i} + 4\mathbf{j} - 1\mathbf{k}$$

is normal to the second surface at the desired point (thus is also normal to that tangent plane).

Now our two tangent planes share both a point (that is, $(0, 2, 2)$) and a normal vector, and hence are the same plane. Note that it would have sufficed for these normal vectors to be parallel (that is, for one to be a scalar multiple of the other) but it happened to work out that they were exactly the same since we picked nice g_1 and g_2.

(b) We wish to investigate the tangent planes of our surface, so just as above, we take

$$g(x, y, z) = x^3 + y - z^2$$

so that

$$\nabla g(x, y, z) = 3x^2\mathbf{i} + 1\mathbf{j} - 2z\mathbf{k}$$

is normal to our surface, as long as we are subject to the constraint $x^3 + y - z^2 = 10$ (or, rearranging, $y = 10 - x^3 + z^2$). We would like this vector to be parallel to the normal vectors of the given plane, for example to $\mathbf{n} = 27\mathbf{i} + 1\mathbf{j} - 8\mathbf{k}$. Thus let

$$\nabla g(x, y, z) = \lambda\mathbf{n}$$

for some $\lambda \neq 0$, subject to the constraint $y = 10 - x^3 + z^2$. The vector equation expands to

(1) $$3x^2 = \lambda 27$$

(2) $$1 = \lambda 1$$

(3) $$-2z = \lambda(-8)$$

and, from (2), we see that $\lambda = 1$. using that to simplify (1) and (3) we obtain $x = \pm 3$ and $z = 4$. Now plugging these values into the constraint, we obtain two solution points,

$$\boxed{(3, -1, 4) \quad \text{and} \quad (-3, 53, 4)}.$$

7. We have

$$f_x = \frac{y}{2\sqrt{1+x}} + \sqrt{1+y}$$

and

$$f_y = \sqrt{1+x} + \frac{x}{2\sqrt{1+y}}$$

which are both continuous for all (x, y) in the domain. Thus the only critical points will be those where both first partials are simultaneously 0, i.e. solutions to the system

$$0 = \frac{y}{2\sqrt{1+x}} + \sqrt{1+y}$$
$$0 = \sqrt{1+x} + \frac{x}{2\sqrt{1+y}}$$

or, rearranging,

$$y = -2\sqrt{1+x}\sqrt{1+y}$$
$$x = -2\sqrt{1+x}\sqrt{1+y}$$

so that first of all $x = y$. Back-substituting,

$$x = -2(1+x)$$

and so

$$x = -2/3,$$

thus our only critical point is $\boxed{\left(-\frac{2}{3}, -\frac{2}{3}\right)}$. We compute the various second derivatives:

$$f_{xx} = \frac{-y}{4(1+x)^{3/2}}$$
$$f_{yy} = \frac{-x}{4(1+y)^{3/2}}$$
$$f_{xy} = f_{yx} = \frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1+y}}$$

and evaluate them at the critical point:

$$f_{xx}\left(\frac{-2}{3}, \frac{-2}{3}\right) = \frac{2/3}{4(1/3)^{3/2}} = \sqrt{3}/2$$
$$f_{yy}\left(\frac{-2}{3}, \frac{-2}{3}\right) = \frac{2/3}{4(1/3)^{3/2}} = \sqrt{3}/2$$
$$f_{xy}\left(\frac{-2}{3}, \frac{-2}{3}\right) = f_{yx}\left(\frac{-2}{3}, \frac{-2}{3}\right) = \frac{1}{2\sqrt{1/3}} + \frac{1}{2\sqrt{1/3}} = \sqrt{3}$$

so that our discriminant is

$$D\left(\frac{-2}{3}, \frac{-2}{3}\right) = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - (\sqrt{3})^2 = \frac{-9}{4} < 0$$

and hence this is a $\boxed{\text{saddle point}}$.

8. (a) We have $\nabla \times \mathbf{F} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (2y-2y)\mathbf{k} = 0$, and $\mathbf{F}$ is defined everywhere since polynomial in each component. So it is conservative.

(b) This integral is path-independent (by part (a)) so we can use the FTOLI. First let's find the potential function. If $\nabla f = \mathbf{F}$, then we have

$$f_x = x^2 + y^2$$
$$f_y = 2xy$$
$$f_z = 3z$$

so that, integrating the first equation, $f = \frac{x^3}{3} + xy^2 + g(y,z)$ and so

$$f_y = 0 + 2xy + g_y(y,z).$$

Hence $g_y = 0$. So g is just a function of z, say $h(z)$. Thus

$$f = \frac{x^3}{3} + xy^2 + h(z)$$

and

$$f_z = 0 + 0 + h'(z)$$

so $h'(z) = 3z$. Thus $h(z) = \frac{3z^2}{2}$ and so

$$f = \frac{x^3}{3} + xy^2 + \frac{3z^2}{2}.$$

Now we may apply the FTOLI:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= f(2,2,0) - f(-2,-2,0) \\ &= \left(\frac{8}{3} + 8 + 0\right) - \left(\frac{-8}{3} - 8 + 0\right) \\ &= \boxed{\frac{64}{3}}. \end{aligned}$$