Math 241 Exam 2 Sample 2 solutions

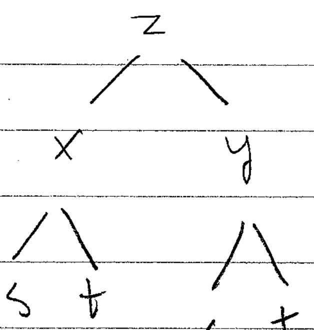
1 (a) $f(x,y) = 3x^2 - 2xy - y$

$\nabla f(x,y) = (6x-2y, -2x-1)$

$\nabla f(2,1) = (10, -5)$

$$D_u f(2,1) = \nabla f(2,1) \cdot u, \text{ as } u \text{ is a unitvector}$$
$$= \cos\theta \cdot \|\nabla f(2,1)\| \, \|u\|$$
$$= \cos 60^\circ \left(10^2 + (-5)^2\right)^{1/2} (1)$$
$$= \frac{1}{2}\sqrt{125} = \boxed{\frac{5}{2}\sqrt{5}}$$

(b) $z = 2x^2 + xy - xy^2$, $x = 2s + t$, $y = e^t$.



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (4x + y - 2xy)(1) + (x - 2xy)(e^t)$$

$$= 4x + y(1-2x) + x(1-2y)e^t$$

$$= \boxed{4(2s+t) + e^t\left(1 - 2(2s+t)\right) + (2s+t)(1-2e^t)e^t}$$

Important: Need to write everything in terms of t — meaning no x's or y's in final answer.

(c) ∇f is orthogonal to the curve.

$f(x,y) = x^2 + 2xy - 3y^2$

$\nabla f(x,y) = (2x + 2y, \; 2x - 6y)$

$0 = \nabla f(x,y) \cdot (2, 1)$

$= 2(2x+2y) + 2x - 6y$

$= 4x + 4y + 2x - 6y$

$= 6x - 2y$

$\Rightarrow \quad 2y = 6x \Rightarrow y = 3x.$

Plug in $-10 = x^2 + 2xy - 3y^2$

$= x^2 + 2x(3x) - 3(3x)^2$

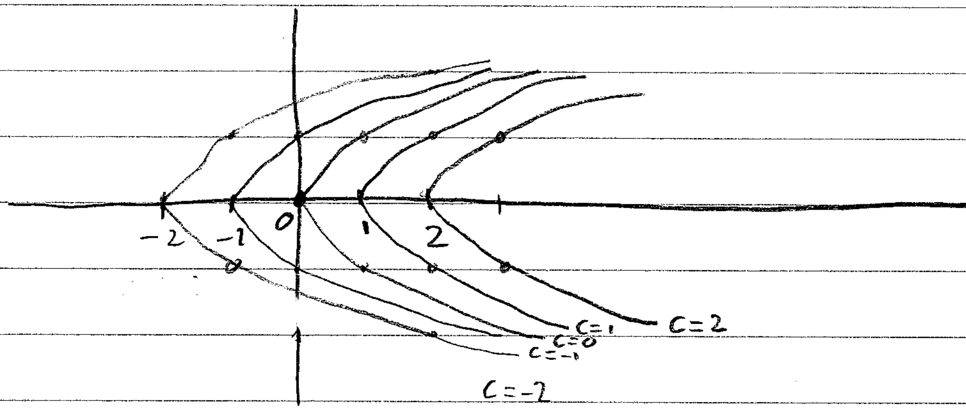
$= x^2 + 6x^2 - 27x^2$

$= -20x^2$

$\Rightarrow \quad 1 = 2x^2 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}. \quad y = \pm \frac{3}{\sqrt{2}}$

Hence $\boxed{\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)}$

2(a) $f(x,y) = x - y^2 = c \Rightarrow x = c + y^2$



(b) $f(x,y,z) = x^2 + 2z^2, \quad c = 22$

$\nabla f(x,y,z) = (2x, 0, 4z)$

$\nabla f(2,7,-3) = (4, 0, -12) = \vec{n}$

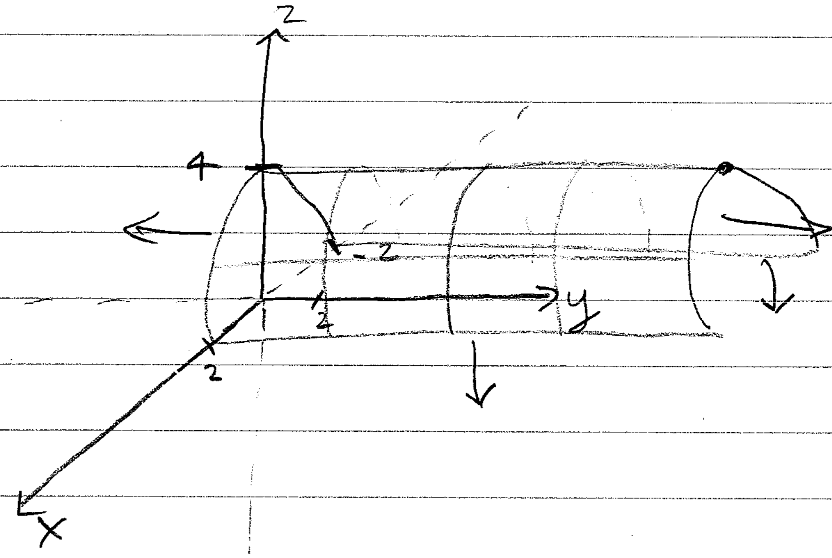
$P = (2, 7, -3)$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

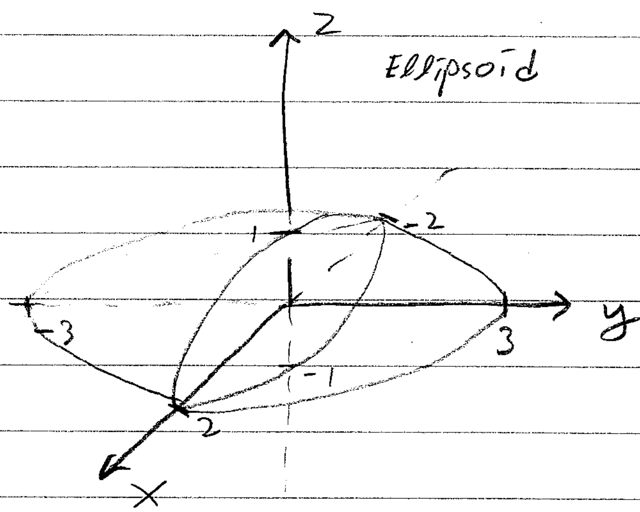
$4(x-2) + 0(y-7) + (-12)(z-(-3)) = 0$

$4(x-2) - 12(z+3) = 0$

3(a) $z = 4 - x^2$ Parabolic Sheet



(b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$ Ellipsoid



x-int $\dfrac{x^2}{4} = 1 \Rightarrow x = \pm 2$

y-int $\dfrac{y^2}{9} = 1 \Rightarrow y = \pm 3$

z-int $z = \pm 1$

(c) $z^2 = x^2 + y^2, \; z \le 0$ cone opening down

$\boxed{(z-5)^2 = x^2 + y^2, \; z \le 0}$ shift up by 5

Alternatively, $z - 5 = -\sqrt{x^2 + y^2}$ (negative b/c opens down)

$$\boxed{z = 5 - \sqrt{x^2 + y^2}}$$

(d) z is free

$(3,0,0)-(0,2,0) = (3,-2,0)$ in plane

$(0,0,1)$ in plane

$$(3i-2j)\times k = \begin{bmatrix} i & j & k \\ 3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= i(-2) - j(3) + k(0) = -2i - 3j = n$$

$P = (3,0,0)$

$-2(x-3) - 3(y-0) + (0)z = 0$ ← This is fine. Don't need to si-plify.

$-2(x-3) - 3y = 0$

$-2x + 6 - 3y = 0$

$-2x - 3y = -6$

$$\boxed{2x + 3y = 6}$$

4)

$$\nabla f(x,y) = \left(2xy - 2y,\ \ x^2 - 2x + 4y - 15\right) = 0$$

$$2xy - 2y = 0 \quad \text{and} \quad \underbrace{x^2 - 2x + 4y - 15 = 0}_{(*)}$$

$\Rightarrow \ y(2x-2) = 0$ (If the product $AB=0$, then $A=0$ or $B=0$.)

$\Rightarrow y = 0$ or $x = 1$

If $y=0$, $(*) \Rightarrow$ $x^2 - 2x - 15 = 0$

$(x-5)(x+3) = 0$

$x = 5, -3$

$(5,0), (-3,0)$ critical pts

If $y \neq 0$, $x = 1$, so $(*) \Rightarrow$ $1 - 2 + 4y - 15 = 0$

$\Rightarrow -1 - 15 + 4y = 0$

$\Rightarrow 4y = 16$

$\Rightarrow y = 4 \Rightarrow (1,4)$ is c.p.

Get $(5,0), (-3,0), (1,4)$.

$f_{xx} = 2y$

$f_{yy} = 4$

$f_{xy} = 2x - 2$

$$D = f_{xx}f_{yy} - f_{xy}^2 = 2y(4) - (2x-2)^2 = 8y - 4(x-1)^2$$

At $(5,0)$ $D = 40 > 0$ and $f_{yy} = 4 > 0 \Rightarrow$ Relative Min

At $(-3,0)$, $D = -24 < 0 \Rightarrow$ Saddle Point

At $(1,4)$, $D = 8 - 4(3)^2 = -28 < 0 \Rightarrow$ Saddle Point.

5) Lagrange Multipliers

$$L = ax + by + \lambda(x^2 + y^2 - r^2)$$

$$\nabla L = (a + 2\lambda x,\ b + 2\lambda y) = 0$$

$$2\lambda x = -a \qquad\qquad 2\lambda y = -b$$

$$\lambda = \frac{-a}{2x} \qquad\qquad \lambda = \frac{-b}{2y}$$

$$\frac{-a}{2x} = \lambda = \frac{-b}{2y}$$

$$\frac{2y}{2x} = \frac{-b}{-a}$$

$$\frac{y}{x} = \frac{b}{a} \Rightarrow y = \frac{b}{a} \cdot x$$

$$r^2 = x^2 + y^2 = x^2 + \left(\frac{b}{a}x\right)^2 = x^2 + x^2\frac{b^2}{a^2}$$

$$= x^2\left(1 + \frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad x^2 = \frac{r^2}{1 + \frac{b^2}{a^2}} \Rightarrow x = \frac{\pm r}{\sqrt{1 + \frac{b^2}{a^2}}}.$$

Critical points $\left(\frac{r}{\sqrt{1+\frac{b^2}{a^2}}},\ \frac{r\frac{b}{a}}{\sqrt{1+\frac{b^2}{a^2}}}\right)$ max

$$y = \frac{\pm\frac{b}{a}\cdot r}{\sqrt{1+\frac{b^2}{a^2}}}$$

$$\left(\frac{-r}{\sqrt{1+\frac{b^2}{a^2}}},\ \frac{-r\frac{b}{a}}{\sqrt{1+\frac{b^2}{a^2}}}\right)$$

Observe by observation, since $f(x,y) = ax + by$ and $a>0, b>0$,

$$f\left(\frac{r}{\sqrt{1+\frac{b^2}{a^2}}}, \frac{r\frac{b}{a}}{\sqrt{1+\frac{b^2}{a^2}}}\right) > 0$$

$$f\left(\frac{-r}{\sqrt{1+\frac{b^2}{a^2}}}, \frac{-r\frac{b}{a}}{\sqrt{1+\frac{b^2}{a^2}}}\right) < 0$$

so the first point is a max and the second a min. Hence

$$\text{Max value} = \frac{ar + \frac{brb}{a}}{\sqrt{1+\frac{b^2}{a^2}}} = \frac{ra\left(1+\frac{b^2}{a^2}\right)}{\sqrt{1+\frac{b^2}{a^2}}}$$

$$= ra\sqrt{1+b^2/a^2} = \boxed{r\sqrt{a^2+b^2}} \text{ Max value}$$

$$\text{Min value} = \frac{-ar - br\frac{b}{a}}{\sqrt{1+\frac{b^2}{a^2}}} = \boxed{-r\sqrt{a^2+b^2}} \text{ Min value}$$