Math 241 Exam 2 Sample 3 Solutions

1. (a) We have $\bar{u} = \frac{1}{\sqrt{2}}\,\hat{\imath} - \frac{1}{\sqrt{2}}\,\hat{\jmath}$ and so

$$D_{\bar{u}}h = \frac{1}{\sqrt{2}}(6x) - \frac{1}{\sqrt{2}}3x^2$$
$$D_{\bar{u}}h(1,1) = \frac{1}{\sqrt{2}}(6) - \frac{1}{\sqrt{2}}3$$

(b) We rewrite the plane as a level surface for a function of three variables and then take the gradient:

$$f(x,y) = x^2 + y^2$$
$$z = x^2 + y^2$$
$$0 = x^2 + y^2 - z$$
$$g(x,y,z) = x^2 + y^2 - z$$
$$\nabla g(x,y,z) = 2x\,\hat{\imath} + 2y\,\hat{\jmath} - \hat{k}$$
$$\nabla g(2,1,5) = 4\,\hat{\imath} + 2\,\hat{\jmath} - \hat{k}$$

We use this vector as the normal vector and the point $(2,1,5)$ to give the equation of the plane

$$4(x-2) + 2(y-1) - 1(z-5) = 0$$

The point $(1,0,-1)$ is on this plane because

$$4(1-2) + 2(0-1) - (-1-5) = 0$$

2. (a) We know $A = \frac{1}{2}bh$. The chain rule tells us

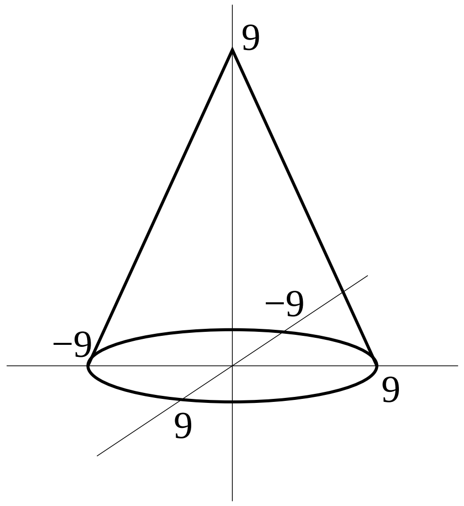
$$\frac{dA}{dt} = \frac{\partial A}{\partial b}\frac{db}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt} = \frac{1}{2}h(2) + \frac{1}{2}b(3)$$

and so at $b = 10$ and $h = 20$ we have

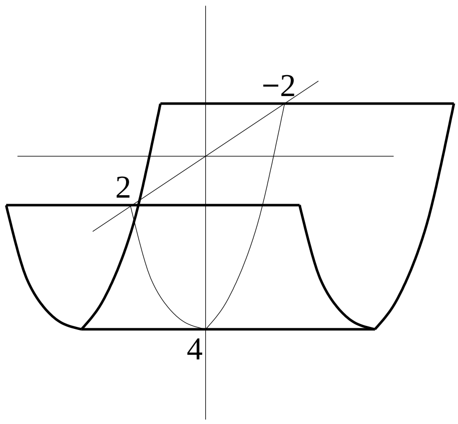
$$\frac{dA}{dt} = \frac{1}{2}(20)(2) + \frac{1}{2}(10)(3)$$

(b) We find $D(x, y) = (2y - 4)(-2) - (2x)^2$ and then test the points:
$D(0, 0) = (-4)(-2) = +$ so then $f_{xx}(0, 0) = -4$ so $(0, 0)$ is a relative maximum.
$D(2, 2) = (0)(-2) - 16 = -$ so $(2, 2)$ is a saddle point.
$D(-2, 2) = (0)(-2) - 16 = -$ so $(-2, 2)$ is a saddle point.

3. (a) We have



(b) We have



(c) $x^2 + y^2 = 9$

(d) $z = 1 + x^2 + y^2$

4. First we find $f_x(x, y) = 2(x - 1) = 0$ when $x = 1$ and $f_y(x, y) = 2y = 0$ when $y = 0$ and the point $(1, 0)$ is in the region so then $f(1, 0) = 0$.

On the boundary:

- For the circular part $y^2 = 4 - x^2$ so $f = (x - 1)^2 + (4 - x^2) = x^2 - 2x + 1 + 4 - x^2 = -2x + 5$ for $0 \leq x \leq 2$ which attains a maximum of 5 (when $x = 0$) and a minimum of 1 (when $x = 2$).

- On the left vertical part $x = 0$ so $f = (0 - 1)^2 + y^2 = y^2 + 1$ for $-2 \leq y \leq 2$ which attains a maximum of 5 (when $y = \pm 2$) and a minimum of 1 (when $y = 0$).

Thus the maximum is 5 and the minimum is 0.

5. The constraint is the level curve for $g(x, y) = x^2 + y^2$ and so we have the system:

$$\begin{aligned} y + 2 &= \lambda(2x) \\ x &= \lambda(2y) \\ x^2 + y^2 &= 4 \end{aligned}$$

Label these (a), (b) and (c).

Then (b) tells us $\lambda = \frac{x}{2y}$ or $y = 0$.
If $y = 0$ then (b) tells us $x = 0$ but (c) tells us $x = \pm 2$ which contradicts itself so $y \neq 0$.
If $\lambda = \frac{x}{2y}$ then into (a) gives us $y + 2 = \left(\frac{x}{2y}\right)(2x)$ so that $x^2 = y(y + 2)$ which goes into (c) to give us

$$\begin{aligned} y(y + 2) + y^2 &= 4 \\ 2y^2 + 2y - 4 &= 0 \\ 2(y + 2)(y - 1) &= 0 \end{aligned}$$

So that $y = -2$ or $y = 1$. If $y = -2$ then (c) tells us $x = 0$ giving us $(0, -2)$ and if $y = 1$ then (c) tells us $x = \pm\sqrt{3}$ giving us $(\pm\sqrt{3}, 1)$.

Then:
$f(0, -2) = 0$
$f(\sqrt{3}, 1) = 3\sqrt{3}$
$f(-\sqrt{3}, 1) = -3\sqrt{3}$
So the maximum is $3\sqrt{3}$ and the minimum is $-3\sqrt{3}$.