Math 241 Sections 01 Exam 2 Solutions**

1. (a) Use the gradient to find a vector perpendicular to the graph of the curve $y = x^3 + x - 2$ at the point where $x = 2$. [10 pts]

   Solution: We write $f(x, y) = x^3 + x - 2 - y$ so then $\nabla f = (3x^2 + 1)\,\hat{\imath} - \hat{\jmath}$. When $x = 2$ we have $y = 8$ and so $\nabla f(2, 8) = 13\,\hat{\imath} - \hat{\jmath}$.

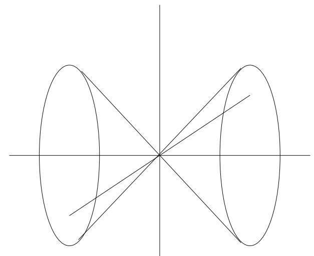
   (b) Suppose the base of a triangle is growing at 2 inches per hour while the height is growing at 3 inches per hour. At what rate is the area growing when the height is 10 inches and the base is 20 inches? [10 pts]

   Solution: We have

$$
\begin{aligned}
A &= \frac{1}{2}bh \\
\frac{dA}{dt} &= \frac{\partial A}{\partial b}\frac{\partial b}{\partial t} + \frac{\partial A}{\partial h}\frac{\partial h}{\partial t} \\
&= \frac{1}{2}h(2) + \frac{1}{2}b(3) \\
\left.\frac{dA}{dt}\right|_{h=10,b=20} &= \frac{1}{2}(10)(2) + \frac{1}{2}(20)(2)
\end{aligned}
$$

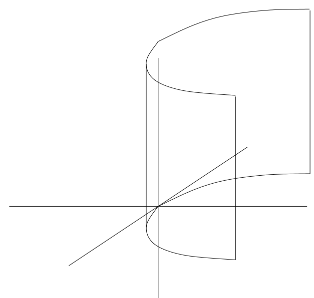
2. (a) Sketch the graph of the surface $y^2 = x^2 + z^2$. Write the name. [5 pts]

Solution: The graph is a double-cone:



(b) Sketch the graph of the surface $y = x^2$. Write the name. [5 pts]

Solution: The graph is a parabolic sheet:



(c) Find the directional derivative of $f(x, y) = y \sin(xy)$ in the direction of $\bar{a} = 2\,\hat{\imath} + \hat{\jmath}$ at the point $\left(\frac{\pi}{8}, 2\right)$. Simplify. [10 pts]

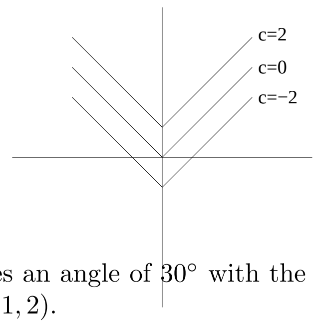
Solution: We have $\bar{u} = \frac{2}{\sqrt{5}}\,\hat{\imath} + \frac{1}{\sqrt{5}}\,\hat{\jmath}$. We have $f_x = y^2 \cos(xy)$ and $f_y = \sin(xy) + xy\cos(xy)$. Then

$$\begin{aligned}
D_{\bar{u}} f\,(x, y) &= \frac{2}{\sqrt{5}}\left(y^2 \cos\left(xy\right)\right) + \frac{1}{\sqrt{5}}\left(\sin\left(xy\right) + xy\cos\left(xy\right)\right) \\
D_{\bar{u}} f\left(\frac{\pi}{8}, 2\right) &= \frac{2}{\sqrt{5}}\left(2^2 \cos\left(\frac{\pi}{8}2\right)\right) + \frac{1}{\sqrt{5}}\left(\sin\left(\frac{\pi}{8}2\right) + \frac{\pi}{8}2\cos\left(\frac{\pi}{8}2\right)\right) \\
&= \frac{2}{\sqrt{5}}\left(4\frac{\sqrt{2}}{2}\right) + \frac{1}{\sqrt{5}}\left(\frac{\sqrt{2}}{2} + \frac{\pi}{8}2\frac{\sqrt{2}}{2}\right)
\end{aligned}$$

3. (a) All together on one graph sketch the level curves for $f(x, y) = y - |x|$ at $c = -2, 0, 2$ and label each with its value of c. [5 pts]

Solution: The functions are:

$$y - |x| = 2y = |x| + 2$$
$$y - |x| = 0y = |x|$$
$$y - |x| = -2y = |x| - 2$$



(b) Suppose the unit vector $\bar{u}$ makes an angle of 30° with the gradient of a function f at $(1, 2)$ and $||\nabla f(1, 2)|| = 3$. Find $D_{\bar{u}} f(1, 2)$. [5 pts]

Solution: We have:

$$\begin{aligned} D_{\bar{u}} f(1, 2) &= \bar{u} \cdot \nabla f(1, 2) \\ &= ||\bar{u}||||\nabla f(1, 2)|| \cos(30°) \\ &= (1)(3)\left(\frac{\sqrt{3}}{2}\right) \end{aligned}$$

(c) The function $f(x, y) = x^2 y - 2x^2 - y^2$ has the following: [10 pts]

$$f_{xx}(x, y) = 2y - 4 \qquad f_{yy}(x, y) = -2 \qquad f_{xy}(x, y) = 2x$$

There are three critical points at $(0, 0)$, $(2, 2)$ and $(-2, 2)$. Categorize each critical point as a relative maximum, relative minimum or saddle point.

Solution: We find $D(x, y) = (2y - 4)(-2) - (2x)^2$ and then test the points:
$D(0, 0) = (-4)(-2) = +$ so then $f_{xx}(0, 0) = -4$ so $(0, 0)$ is a relative maximum.
$D(2, 2) = (0)(-2) - 16 = -$ so $(2, 2)$ is a saddle point.
$D(-2, 2) = (0)(-2) - 16 = -$ so $(-2, 2)$ is a saddle point.

4. Find the maximum and minimum values of $f(x,y) = x^2 + 2y^2$ on the quarter circle $x^2 + y^2 \leq 4$ with $x, y \geq 0$. [20 pts]

Solution: First we check the critical points. We have $f_x = 2x$ and $f_y = 4y$. When these equal zero we have $(0,0)$ and $f(0,0) = 0$.

Then we check the edge:
Left side: Here $x = 0$ so $f = 2y^2$ with $0 \leq y \leq 2$ which has a minimum of 0 and a maximum of 8.
Bottom side: Here $y = 0$ so $f = x^2$ with $0 \leq x \leq 2$ which has a minimum of 0 and a maximum of 4.
Round side: Here $y^2 = 4 - x^2$ so $f = x^2 + 2(4 - x^2) = -x^2 + 8$ with $0 \leq x \leq 2$ which has a minimum of 4 and a maximum of 8.

Thus overall the maximum is 8 and the minimum is 0.

5. Let $f(x, y) = x^2 + 6y^2$ and suppose (x, y) is constrained by $x + 3y = 10$.

(a) Use Lagrange multipliers to find the minimum of $f(x, y)$ subject to the constraint. [16 pts]

Solution: We have

$$\text{Objective: } f(x, y) = x^2 + 6y^2$$
$$\text{Constraint: } g(x, y) = x + 3y = 10$$

and so our system of equations is

$$2x = \lambda$$
$$12y = \lambda(3)$$
$$x + 3y = 10$$

The first gives us $\lambda = 2x$ and the second gives us $\lambda = 4y$. thus $2x = 4y$ and $x = 2y$. Plugging this into the third gives $2y + 3y = 10$ so $y = 2$ and $x = 4$. Thus we have $(2, 4)$ and $f(2, 4) = 100$.

(b) Explain why $f(x, y)$ has no maximum subject to the constraint. [4 pts]

Solution: Basically we can make x very positive and y very negative, keeping $x + 3y = 10$, but then f is arbitrarily large.