Math 241 Exam 4 Sample 2

Directions: Do not simplify or evaluated unless indicated. No calculators are permitted. Show all work as appropriate for the methods taught in this course. Partial credit will be given for any work, words or ideas which are relevant to the problem.

Please put problem 1 on answer sheet 1

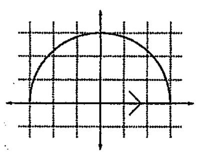
1. (a) Suppose $\vec{F}(x,y,z) = (6xy + z\cos(x))\,\hat{\imath} + (3x^2)\,\hat{\jmath} + (\sin(x) - 1)\,\hat{k}$. Using that systematic method shown in class, find a function $f(x,y,z)$ such that $\vec{F} = \nabla f$. [10 pts]

   (b) First explain why the integral $\int_C \frac{y}{z}\,dx + \frac{x}{z}\,dy - \frac{xy}{z^2}\,dz$ for $z > 0$ is independent of path and then evaluate this integral where C is any curve from $(1,2,1)$ to $(0,3,3)$. [10 pts]

Please put problem 2 on answer sheet 2

2. (a) Evaluate the integral $\int_C x + \frac{16}{3}y - z + 8\; ds$, where C is the curve with parametrization $\vec{r}(t) = t^2\,\hat{\imath} + 3t\,\hat{\jmath} + (t^2+8)\,\hat{k}$ for $0 \le t \le 2$. [10 pts]

   (b) Use Green's Theorem to evaluate $\int_C 3y\;dx + (2xy+4)\;dy$, where C is as shown: [10 pts]



Please put problem 3 on answer sheet 3

3. Let Σ be the part of the parabolic sheet $z = 4 - x^2$ above the xy plane and between $y = 0$ and $y = 4$ with downwards orientation. Draw a picture of Σ and evaluate the integral $\iint_\Sigma \vec{F}\cdot\vec{n}\;dS$, where $\vec{F}(x,y,z) = 3\,\hat{\imath} + 2z\,\hat{\jmath} + z\,\hat{k}$. [20 pts]

Please put problem 4 on answer sheet 4

4. Let C be the intersection of the cylinder $x^2 + y^2 = 9$ with the parabolic sheet $z = y^2$. [20 pts]
   Suppose C has the clockwise orientation when viewed from above. Use Stokes' Theorem to convert $\int_C (4x^2z^2\,\hat{\imath} + xy\,\hat{\jmath} + y^3\,\hat{k})\cdot d\vec{r}$ to a surface integral. Give an explicit description of your surface Σ as the graph of a function f on a region R, then rewrite your surface integral as an iterated integral in whichever coordinate system (polar or rectangular) you find most appropriate. You do not need to evaluate the final integral.

Please put problem 5 on answer sheet 5

5. Suppose Σ is composed of the portion of $x^2 + z^2 = 4$ between $y = -a$ and $y = a$, along with the disks of radius 2 which seal the cylinder on each end. Suppose a fluid flow is given by $\vec{F}(x,y,z) = 3xy^2\,\hat{\imath} - y^3\,\hat{\jmath} + 4z\,\hat{k}$. Use the Divergence Theorem to find the appropriate a such that the fluid is flowing out through Σ at a rate of 128π. [20 pts]

Math 241 Exam 4 Sample 2 Solutions

1 (a)

$f_x = 6xy + z\cos(x)$ (1)

$f_y = 3x^2$ (2)

$f_z = \sin(x) - 1$ (3)

$f(x,y,z) = \int 6xy + z\cos(x)\,dx$

$= \frac{6x^2y}{2} + z\sin(x) + g(y,z)$

$f_y(x) = \frac{\partial}{\partial y}\left(3x^2y + z\sin(x) + g(y,z)\right)$

$= 3x^2 + g_y$

Compare to (2) and conclude $g_y(y,z) = 0$.

$g(y,z) = \int 0\,dy = h(z)$

Hence $f(x,y,z) = 3x^2 + z\sin(x) + h(z)$

$f_z = \sin(x) + h_z$

Compare to (3) and conclude $h_z = -1$

$h(z) = \int -1\,dz = -z + C$

Hence $f(x,y,z) = \boxed{3x^2y + z\sin(x) - z + C}$

It's a good idea to check your answer by computing ∇f. □

1(b) $F(x,y,z) = \frac{y}{z}\vec{i} + \frac{x}{z}\vec{j} - \frac{xy}{z^2}\vec{k}$

$$\operatorname{curl} F = \nabla \times F$$

$$= \begin{bmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z} \end{bmatrix}$$

$$= \left(\frac{1}{z} - \frac{1}{z}\right)\vec{i} - \left(\frac{-y}{z} - \left(-\frac{y}{z^2}\right)\right)\vec{j}$$

$$+ \left(\frac{1}{z} - \frac{1}{z}\right)\vec{k} = 0$$

Since $\operatorname{curl} F = 0$ and $F(x,y,z)$ is defined for $z>0$, F is conservative in the region $z>0$. Hence the integral is independent of path by the fundamental theorem of line integrals (FTOLI).

Observe $f(x,y,z) = \frac{xy}{z}$ is a potential for $\vec{F}(x,y,z)$. Since both points satisfy $z>0$, by FTOLI

$$\int_C \vec{F}\cdot d\vec{r} = f(\text{endpoint}) - f(\text{start point})$$

$$= f(0,3,3) - f(1,2,1)$$

$$= \frac{0\cdot 3}{3} - \frac{1\cdot 2}{1} = \boxed{-2}.$$

D

2(a)

$$I = \int_C x + \frac{16}{3}y - z + 8\, ds$$

$$r(t) = t^2 i + 3t j + (t^2+8)k$$

$$r'(t) = 2t i + 3 j + 2t k$$

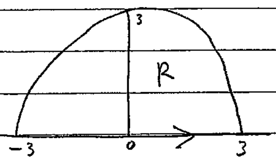
$$\|r'(t)\| = \sqrt{8t^2+9}$$

$$I = \int_0^2 \left(\cancel{t^2} + \frac{16}{\cancel{3}}(\cancel{3}t) - (\cancel{t^2}+8)\right) + 8\sqrt{8t^2+9}\, dt$$

$$= \int_0^2 (16t - \cancel{8})\sqrt{8t^2+9}\, dt = \int_0^2 16t\sqrt{8t^2+9}\, dt$$

$$= \boxed{\frac{2}{3}(8t^2+9)^{3/2}\Big|_0^2 = \frac{2}{3}(32+9)^{3/2} - \frac{2}{3}(9)^{3/2} = \frac{2}{3}(41)^{3/2} - \frac{2}{3}(27)}$$

(b) $\int_C 3y\,dx + (2xy+4)\,dy$



$M = 3y \qquad N = 2xy + 4$

Since C is counterclockwise, and closed,

$$\int_C M\,dx + N\,dy = \iint_R N_x - M_y \, dA$$

$$= \iint (2y - 3)\, dA$$

$$= \int_0^{\pi} \int_0^{3} (2r\sin\theta - 3)\, r\, dr\, d\theta$$

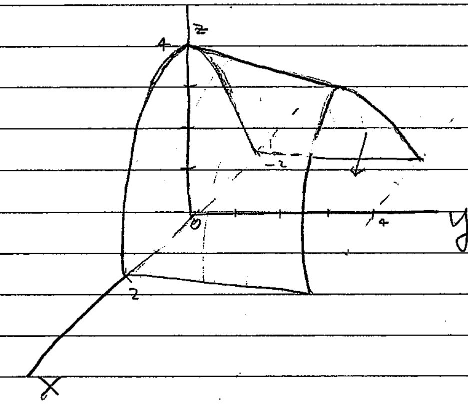
$$= \int_0^{\pi} \frac{2r^3}{3}\sin\theta - \frac{3r^2}{2} \Big|_{r=0}^{3} d\theta$$

$$= \int_0^{\pi} \left(18\sin\theta - \frac{27}{2}\right) d\theta = -18\cos\theta - \frac{27}{2}\theta \Big|_0^{\pi}$$

$$= -18(-1) - \frac{27}{2}\pi - \left(-18(1) - \frac{27}{2}(0)\right)$$

$$= 18 - \frac{27}{2}\pi + 18 = \boxed{36 - \frac{27}{2}\pi}$$

3



$\vec{F}(x,y,z) = 3i + 2z\,j + z\,\vec{k}$

$r(x,y) = xi + yj + (4-x^2)k$

$r_x = i - 2x\,k$

$r_y = j$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & 0 \end{vmatrix} = 2x\,\vec{i} + \vec{k}$$

Since the z-component of $2xi + k$ is positive, the vector $r_x \times r_y$ points up and disagrees with the given downward orientation, so we use a $-$ sign.

$$\int_\Sigma \vec{F}\cdot\vec{n}\,dS = -\int_R \vec{F}\cdot(r_x\times r_y)\,dA$$

$$= -\int_0^2\int_0^4 (3\vec{i} + 2z\vec{j} + (4-x^2)\vec{k})\cdot(2x\vec{i}+\vec{k})\,dy\,dx$$

$$= -\int_0^2\int_0^4 6x + (4-x^2)\;dy\,dx$$

$$= -\int_0^2 4\left(-x^2+6x+4\right)dx$$

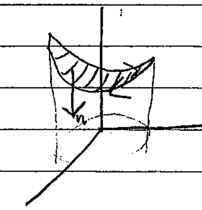
$$= \int_0^2 4x^2 - 24x - 16\,dx$$

$$= \frac{4x^3}{3} - \frac{24x^2}{2} - 16x\,\Big|_0^2$$

$$= \frac{4(2)^3}{3} - \frac{24(2)^2}{2} - 16(2) - 0$$

$$= \frac{32}{3} - 48 - 32 = \boxed{-\frac{208}{3}}$$

4) $$\int_C 4x^2z^2\,i + xy\,j + y^3\,k\cdot dr$$



$$= \pm\iint_\Sigma (\nabla\times\vec{F})\cdot\vec{n}\,dS$$

The orientation that induces the given boundary orientation points down.

$$\vec{F} = 4x^2z^2\,i + xy\,j + y^3\,k$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 4x^2z^2 & xy & y^3 \end{vmatrix}$$

$$= 3y^2\vec{i} + 8x^2z\vec{j} + y\vec{k}$$

$$r(x,y) = x\,i + y\,j + y^2\,k \qquad x^2+y^2\le 1$$

$$r_x = \vec{i}$$
$$r_y = \vec{j} + 2y\vec{k}$$

$$r_x\times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 2y \end{vmatrix} = -2y\vec{j} + \vec{k}$$

Since $r_x\times r_y$ points up, it disagrees with the given orientati
so use $-$.

$$= -\iint_{x^2+y^2\le 1} \left(3y^2 \vec{i} + 8x^2 z \vec{j} + y \vec{k}\right)\left(-2y \vec{j} + \vec{k}\right) dA$$

$$= -\iint -16x^2 y z + y \quad dA$$

$$= -\iint -16\, x^2\, y(y^2) + y \; dA$$

$$= \iint 16x^2y^3 + y \; dA$$

$$= \boxed{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 16x^2 y^3 + y \; dy\, dx}$$

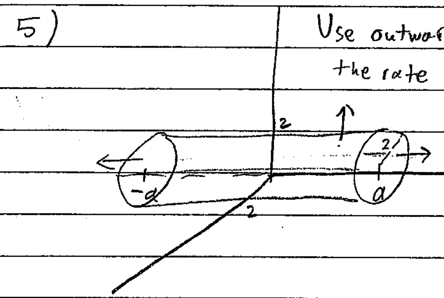
Stop here. Do not evaluate.

Just to see: $= \displaystyle\int_{-1}^{1} \left. 16x^2 \frac{y^4}{4} + y^2 \right|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$

$$= \int \left(16x^2 \frac{(1-x^2)^2}{4} + (1-x^2)^2\right)2 \; dx$$

Note: Even though the integral is on a circle, evaluating the integral as vertically simple is easier than as polar.

5) Use outward normal, since we are measuring the rate at which fluid flows out.



We want to find the value of a such that

$$128\pi = \iint_\Sigma \vec{F}\cdot n \, dS.$$

By the divergence theorem, this integral is

$$= \iiint_D \operatorname{div} \vec{F}\, dV$$

$$F(x,y,z) = 3xy^2\vec{i} - y^3\vec{j} + 4z\vec{k}$$

$$\operatorname{div} F = \nabla \cdot F = 3y^2 - 3y^2 + 4 = 4$$

Hence $128\pi = \iiint_D \operatorname{div} \vec{F}\, dV = 4\iiint_D dV$

$= 4 \cdot$ volume of cylinder.

The volume of the cylinder is area of base $\times$ height

$= \pi(2)^2(2a) = 8\pi a$

Hence $128\,\cancel{\pi} = 4 \cdot \text{volume} = 32\,\cancel{\pi} a \Rightarrow \boxed{a = 4}$