Math 241 Exam 4 Sample 3 Solutions

1. We're looking for

$$\iint_{\Sigma} (0\,\hat{\imath} + x\,\hat{\jmath} + z\,\hat{k}) \cdot \bar{n}\ dS$$

We parametrize Σ by

$$\bar{r}(x, y) = x\,\hat{\imath} + y\,\hat{\jmath} + (9 - x^2)\,\hat{k} \quad \text{with} \quad 0 \le x \le 2,\ 0 \le y \le 2$$

and then

$$\begin{aligned} \bar{r}_x &= 1\,\hat{\imath} + 0\,\hat{\jmath} - 2x\,\hat{k} \\ \bar{r}_y &= 0\,\hat{\imath} + 1\,\hat{\jmath} + 0\,\hat{k} \\ \bar{r}_x \times \bar{r}_y &= 2x\,\hat{\imath} + 0\,\hat{\jmath} + 1\,\hat{k} \end{aligned}$$

Note that these vectors have a positive $\hat{k}$ component which matches the orientation of Σ. So we have

$$\begin{aligned} \iint_{\Sigma} (0\,\hat{\imath} + x\,\hat{\jmath} + z\,\hat{k}) \cdot \bar{n}\ dS &= +\iint_{R} (0\,\hat{\imath} + x\,\hat{\jmath} + (9 - x^2)\,\hat{k}) \cdot (2x\,\hat{\imath} + 0\,\hat{\jmath} + 1\,\hat{k})\ dA \\ &= \iint_{R} 9 - x^2\ dA \\ &= \int_0^2 \int_0^2 9 - x^2\ dy\ dx \\ &= \int_0^2 9y - x^2 y \Big|_0^2\ dx \\ &= \int_0^2 18 - 2x^2\ dx \\ &= 18x - \frac{2}{3}x^3 \Big|_0^2 \\ &= 18(2) - \frac{2}{3}(2)^3 \end{aligned}$$

2. (a) Since $\bar{F}$ is conservative with potential function $f(x, y) = \frac{1}{2}x^2y^2 + x$ and so

$$\begin{aligned}\int_C x^2y + 1 \ dx + xy^2 \ dy &= f(3,3) - f(1,-2) \\ &= \left[\frac{1}{2}(3)^2(3)^2 + 3\right] - \left[\frac{1}{2}(1)^2(-2)^2 + 1\right]\end{aligned}$$

(b) We parametrize the line segment as

$$\bar{r}(t) = 5t\,\hat{\imath} + 4t\,\hat{\jmath} \quad \text{with} \quad 0 \leq t \leq 1$$

and then

$$\begin{aligned}\bar{r}'(t) &= 5\,\hat{\imath} + 4\,\hat{\jmath} \\ ||\bar{r}'(t)|| &= \sqrt{41}\end{aligned}$$

and then

$$\int_C 2x + y \ ds = \int_0^1 [2(5t) + 4t]\,\sqrt{41}\ dt$$

3. By Green's Theorem we have

$$\int_C 2x\ dx + x^2\ dy = \iint_R 2x - 0\ dA$$

where R is the region inside the curve. This region is parametrized best in polar coordinates so we have

$$\begin{aligned}
\int_C 2x\ dx + x^2\ dy &= \iint_R 2x - 0\ dA \\
&= \int_0^{\pi/2} \int_1^2 2r\cos\theta\ r\ drd\theta \\
&= \int_0^{\pi/2} \frac{2}{3} r^3 \cos\theta \Big|_1^2\ d\theta \\
&= \int_0^{\pi/2} \frac{14}{3} \cos\theta\ d\theta \\
&= \frac{14}{3} \sin\theta \Big|_0^{\pi/2} \\
&= \frac{14}{3} [\sin(\pi/2) - \sin(0)] \\
&= \frac{14}{3}
\end{aligned}$$

4. The best Σ would be the portion of the plane $x + y = 5$ inside the cylinder. The orientation of Σ would be toward the right.

   Since $\bar{F}(x, y, z) = x\,\hat{\imath} + 3\,\hat{\jmath} + 2y\,\hat{k}$ we have $\nabla \times F = 2\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}$.

   Then Stokes's Theorem tells us

$$\int_C x\ dx + 3\ dy + 2y\ dz = \iint_\Sigma (2\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}) \cdot \bar{n}\ dS$$

   Since Σ is inside the cylinder it's a good choice to parametrize it as

$$\bar{r}(r, \theta) = r\cos\theta\,\hat{\imath} + (5 - r\cos\theta)\,\hat{\jmath} + r\sin\theta\,\hat{k} \quad \text{with} \quad 0 \le r \le 2,\ 0 \le \theta \le 2\pi$$

   Then

$$\begin{aligned} \bar{r}_r &= \cos\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath} + \sin\theta\,\hat{k} \\ \bar{r}_\theta &= -r\sin\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r\cos\theta\,\hat{k} \\ \bar{r}_r \times \bar{r}_\theta &= -r\,\hat{\imath} - r\,\hat{\jmath} + 0\,\hat{k} \end{aligned}$$

   Note that these vectors point have negative $\hat{k}$ component and hence point left, opposite to that for Σ. So we have

$$\begin{aligned} \iint_\Sigma (2\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}) \cdot \bar{n}\ dS &= -\iint_R (2\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k}) \cdot (-r\,\hat{\imath} - r\,\hat{\jmath} + 0\,\hat{k})\ dA \\ &= -\iint_R -2r\ dA \\ &= -\int_0^{2\pi}\int_0^2 -2r\ dr\ d\theta \end{aligned}$$

5. If D is the solid cube then the Divergence Theorem gives us

$$\begin{aligned}\iint_{\Sigma} (5x\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{k}) \cdot \bar{n}\ dS &= \iiint_{D} (5 + 2 - 2)\ dV \\ &= 5 \iiint_{D} 1\ dV \\ &= 5 \text{ Volume of Cube} \\ &= 5(8) \\ &= 40\end{aligned}$$