Math 241 Parametrization of Surfaces

First make sure that you understand what a parametrization of a surface Σ actually means. To say that Σ is parametrized by $\bar{r}(u,v) = x(u,v)\,\hat{\imath} + y(u,v)\,\hat{\jmath} + z(u,v)\,\hat{k}$ for all u, v within the region R in the uv-plane means that if you take all possible u and v with in your region R then you get the entire surface with the resulting points $(x(u,v), y(u,v), z(u,v))$. In other words think of the vectors $\bar{r}(u,v)$ as just being points.

For example, consider the parametrization $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + 2\,\hat{k}$ with $0 \le x \le 2$ and $0 \le y \le 3$. As x varies and y varies within their allowable ranges we get all the points $(x, y, 2)$ with $0 \le x \le 2$ and $0 \le y \le 3$. This gives us a small rectangular piece of the plane $z = 2$.

This is a very simple example but is a good start. Here are a series of ideas you can consider when presented with a description of Σ. Following each are some problems which fit that criteria. Some have solutions, some have hints, some have notes.

1. Is Σ a part of the graph of a function $z = f(x,y)$ defined on some x, y which are themselves nicely parametrized by rectangular coordinates? If so then we can use

   $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + f(x,y)\,\hat{k}$ with R the region of allowable x and y.

   (a) **Example:** Σ is the part of the cone $z = \sqrt{x^2 + y^2}$ above the rectangle in the xy-plane with opposite corners $(1,0)$ and $(2,5)$.

   Solution: $\bar{r}(x,y) = x\,\hat{\imath} + y\,\hat{\jmath} + \sqrt{x^2 + y^2}\,\hat{k}$ with $1 \le x \le 2$ and $0 \le y \le 5$.

   (b) **Example:** Σ is the part of the paraboloid $z = 9 - x^2 - y^2$ above the triangle in the xy-plane with corners $(0,0)$, $(4,0)$ and $(0,2)$.

   (c) **Example:** Σ is the part of the plane $z = 20 - x - 2y$ above R, where R is the region in the xy-plane between $y = x^2$ and $y = 4$.

   Hint: You'll need to parametrize R as vertically simple.

2. Is Σ a part of the graph of a function $z = f(x,y)$ defined on some x, y which are themselves nicely parametrized by polar coordinates? If so then we can use

   $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + f(r\cos\theta, r\sin\theta)\,\hat{k}$, with R the region of allowable r and θ.

   Try not to think of r and θ as polar coordinates here though, just think of them as variables with a certain range and as they vary over that range the function $\bar{r}(r,\theta)$ gives all the points on the surface. For example if your parametrization for some problem turned out to be $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r^3\,\hat{k}$ for $0 \le \theta \le \pi$ and $0 \le r \le \sin\theta$ then you could just as readily use any variables, for example $\bar{r}(t,q) = t\cos q\,\hat{\imath} + t\sin q\,\hat{\jmath} + t^3\,\hat{k}$ for $0 \le q \le \pi$ and $0 \le t \le \sin q$. No difference. You're just using what you know about polar coordinates to come up with the parametrization.

   (a) **Example:** Σ is the part of the cone $z = 2 + \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$.

   (b) **Example:** Σ is the part of the parabolic sheet $z = y^2$ inside the cylinder $r = \sin\theta$.

   Solution: $\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + r^2\sin^2\theta\,\hat{k}$ for $0 \le \theta \le \pi$ and $0 \le r \le \sin\theta$.

   (c) **Example:** Σ is the part of the plane $z = 20 - x - 2y$ in the first octant and inside $r = 2$.

3. In some cases the above two situations can also work with the variables switched around in the cases where Σ is part of a surface given by $x = f(y, z)$ or $y = f(x, z)$. This is rare but it's useful to work some out.

   (a) **Example:** Σ is the part of the paraboloid $y = x^2 + z^2$ to the right of the square in the xz-plane with corners $(0, 0)$, $(2, 0)$, $(0, 2)$ and $(2, 2)$.

   Hint: Your two variables will be x and z. The region R will be in the xz-plane and y will depend upon x and z.

   (b) **Example:** Σ is the part of the parabolic sheet $x = 16 - z^2$ inside the cylinder $y^2 + z^2 = 9$.

   Hint: Since x depends on z and since y and z always lie within a circle we should use what we know about polar coordinates but with the variables switched. Try using $y = r\cos\theta$ and $z = r\sin\theta$. What would x be? How would R be described and in what plane?

4. If none of these are the case then we need to custom-design a parametrization based upon the surface in question. It may also be the case that a problem can be done in one of the previous ways but it simply works out better this way.

   (a) **Example:** Σ is the part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$.

   Hint: z is free to vary between 0 and 2 independent of x and y so it should be its own variable. Can x and y both be determined by some other variable, perhaps θ?

   (b) **Example:** Σ is the part of the cylinder $x^2 + z^2 = 9$ between $y = 0$ and $y = 2$.

   Hint: Tweak the previous example.

   (c) **Example:** Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ below the cone $z = \sqrt{x^2 + y^2}$.

   Hint: Your knowledge of spherical coordinates should give you a parametrization $\bar{r}(\phi, \theta)$.

   (d) **Example:** Σ is the part of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 2$ and in the first octant.

   Note: We could treat this part of the cylinder as $y = \sqrt{9 - x^2}$ then do $\bar{r}(x, z) = x\,\hat{\imath} + \sqrt{9 - x^2}\,\hat{\jmath} + z\,\hat{k}$ for $0 \le x \le 3$ and $0 \le z \le 2$ but this is not so pretty.

   Instead how about $\bar{r}(z, \theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + z\,\hat{k}$ for $0 \le \theta \le \pi/2$ and $0 \le z \le 2$.

   (e) **Example:** Σ is the part of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane.

   Note: This can be done solving for z and treating it as function of x and y and using polar but it's certainly much easier using spherical coordinates to get a parametrization.