The assertion that a closed bounded subset of $\mathbb{R}$ is compact is often referred to as the Heine–Borel Theorem. The assertion that a closed bounded subset of $\mathbb{R}$ is sequentially compact is often referred to as the Bolzano–Weierstrass Theorem.

EXERCISES FOR SECTION 2.5

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. Every bounded set is closed.
   b. Every closed set is bounded.
   c. Every closed set is compact.
   d. Every bounded set is compact.
   e. A subset of a compact set is also compact.
2. For each of the following statements, determine whether it is true or false and justify your answer.
   a. The set of irrational numbers is closed.
   b. The set of rational numbers in the interval $[0, 1]$ is compact.
   c. The set of negative numbers is closed.
3. Let a and b be numbers with $a < b$. Define $S \equiv [a, b) \equiv \{x \mid a \leq x < b\}$.
   a. Using the definition of sequential compactness, show that S is not sequentially compact.
   b. Using the definition of compactness, show that S is not compact.
   c. Using the definition of closedness, show that S is not closed.
4. Let S be the set of rational numbers in the interval $[0, 2]$.
   a. Using the definition of sequential compactness, show that S is not sequentially compact.
   b. Using the definition of compactness, show that S is not compact.
   c. Using the definition of closedness, show that S is not closed.
5. Let S be a set consisting of a single point. Show that S is compact.
6. Let $S = [0, 1] \cup [3, 4]$. Show that the set S is compact.
7. Let A and B be compact sets. Show that the union $A \cup B$ and the intersection $A \cap B$ are also compact.
8. Let A and B be sets in $\mathbb{R}$. If the union $A \cup B$ is compact, is it true that both A and B must also be compact?
9. At what single point in the proof that sequential compactness implies compactness is the assumption used that the members of the cover are open intervals?
10. For each natural number n, let I_n be a closed bounded interval. Suppose that $\{I_n\}_{n=1}^{\infty}$ covers the compact set consisting of the closed bounded interval $[0, 1]$. Is it true that this cover has a finite subcover?
11. Examine the proof of the theorem that sequential compactness implies compactness and show that the only property of the sets I_n in the cover that we used was that if a point x lies in I_n, then there is an open interval J centered at the point that also lies in I_n. A set having this property is called *open*.
12. Provide a direct proof that a sequentially compact set must be both closed and

CHAPTER 3

CONTINUOUS FUNCTIONS

3.1 CONTINUITY

In Chapter 2, we considered real-valued functions that have as their domains the set of natural numbers; that is, we considered sequences of real numbers. We now begin the study of real-valued functions having as their domains a general subset of $\mathbb{R}$. There is a standard notation: For a set of real numbers D, by

$$f: D \to \mathbb{R}$$

we denote a function whose domain is D, and for each point x in D we denote by $f(x)$ the value that the function assigns to x. When we write $f: D \to \mathbb{R}$, we will assume without further mention that D is a set of real numbers.

Two of the concepts essential to an analytic description of functions $f: D \to \mathbb{R}$ are *continuity* and *differentiability*. The first five sections of this chapter are devoted to the study of continuity. In the final section we study limits in preparation for the discussion of differentiability, which we will begin in Chapter 4.

Definition A function $f: D \to \mathbb{R}$ is said to be *continuous at the point* x_0 in D provided that whenever $\{x_n\}$ is a sequence in D that converges to x_0, the image sequence $\{f(x_n)\}$ converges to $f(x_0)$. The function $f: D \to \mathbb{R}$ is said to be *continuous* provided that it is continuous at every point in D.

The definition of continuity of the function $f: D \to \mathbb{R}$ at the point x_0 in D is formulated to make precise the intuitive notion that "if x is a point in D that is close to x_0, then its image $f(x)$ is close to $f(x_0)$," or, what is supposed to describe the same property, "the difference $f(x) - f(x_0)$ becomes arbitrarily small if the point x in D is sufficiently close to x_0." These statements are placed in quotation marks because we are unable to make mathematically precise the concepts of "arbitrarily small" and "close." In Section 4, we will consider a different approach to capturing the concept of continuity.

Three Examples

Example 3.1 For each number x, define $f(x) = x^2 - 2x + 4$. Then the function $f: \mathbb{R} \to \mathbb{R}$ is continuous. To verify this, we select a point x_0 in $\mathbb{R}$, and we will show that the function is continuous at x_0. Let $\{x_n\}$ be a sequence that converges to x_0. By the sum and product properties of convergent sequences,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \left[x_n^2 - 2x_n + 4\right] = x_0^2 - 2x_0 + 4 = f(x_0).$$

Thus, f is continuous at x_0. ■

The above example is a special case of the continuity of polynomials. The Polynomial Property of convergent sequences stated in Section 2.1 is a statement of the continuity of polynomials.

Example 3.2 Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0. \end{cases}$$

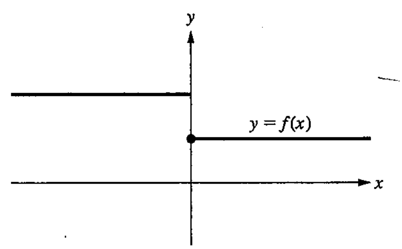

FIGURE 3.1 The function $f: \mathbb{R} \to \mathbb{R}$ is not continuous at $x = 0$.

The function $f: \mathbb{R} \to \mathbb{R}$ is continuous at each point x_0 except for $x_0 = 0$. First consider $x_0 = 0$. The sequence ($\{-1/n\}$ converges to 0. But $\{f(-1/n)\}$ is a constant sequence having all terms equal to 2. Thus,

$$\lim_{n\to\infty} f(-1/n) = 2 \neq 1 = f(0),$$

and so f is not continuous at $x_0 = 0$. Now consider $x_0 \neq 0$. If a sequence $\{x_n\}$ converges to x_0, then there is an index N such that

$$f(x_n) = f(x_0) \qquad \text{for all indices } n \geq N.$$

Thus,

$$\lim_{n\to\infty} f(x_n) = f(x_0),$$

and so f is continuous at the point x_0. ■

Example 3.3 Define the function $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is called *Dirichlet's function*. There is no point x_0 in $\mathbb{R}$ at which Dirichlet's function is continuous. Indeed, given a point x_0 in $\mathbb{R}$, by the sequential density of the rationals and irrationals (recall Theorem 2.20), there is a sequence $\{u_n\}$ of rational numbers that converges to x_0 and also a sequence $\{v_n\}$ of irrational numbers that converges to x_0. But $\{f(u_n)\}$ is a constant sequence all of whose terms equal 1 while $\{f(v_n)\}$ is a constant sequence all of whose terms equal 0. Thus,

$$\lim_{n\to\infty} f(u_n) = 1 \neq 0 = \lim_{n\to\infty} f(v_n).$$

Since both of the sequences $\{u_n\}$ and $\{v_n\}$ converge to x_0, it is not possible for f to be continuous at x_0. Observe that one expression of the discontinuous nature of Dirichlet's function is that there is no way to graph it. ■

Sums, Products, and Quotients of Continuous Functions

Given two functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$, we define the *sum* $f + g: D \to \mathbb{R}$ and the *product* $fg: D \to \mathbb{R}$ by

$$(f+g)(x) \equiv f(x) + g(x) \qquad \text{and} \qquad (fg)(x) \equiv f(x)g(x) \qquad \text{for all } x \text{ in } D.$$

Moreover, if $g(x) \neq 0$ for all x in D, the *quotient* $f/g: D \to \mathbb{R}$ is defined by

$$(f/g)(x) \equiv \frac{f(x)}{g(x)} \qquad \text{for all } x \text{ in } D.$$

The following theorem is an analog, and also a consequence, of the sum, product, and quotient properties of convergent sequences.

Theorem 3.4 Suppose that the functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are continuous at the point x_0 in D. Then the sum

$$f + g: D \to \mathbb{R} \text{ is continuous at } x_0, \tag{3.1}$$

the product

$$fg: D \to \mathbb{R} \text{ is continuous at } x_0, \tag{3.2}$$

and, if $g(x) \neq 0$ for all x in D, the quotient

$$f/g: D \to \mathbb{R} \text{ is continuous at } x_0. \tag{3.3}$$

Proof

Let $\{x_n\}$ be a sequence in D that converges to x_0. By the definition of continuity,

$$\lim_{n\to\infty} f(x_n) = f(x_0) \qquad \text{and} \qquad \lim_{n\to\infty} g(x_n) = g(x_0).$$

The sum property of convergent sequences implies that

$$\lim_{n\to\infty}[f(x_n)+g(x_n)] = f(x_0)+g(x_0), \tag{3.4}$$

and the product property of convergent sequences implies that

$$\lim_{n\to\infty}[f(x_n)g(x_n)] = f(x_0)g(x_0). \tag{3.5}$$

If $g(x) \neq 0$ for all x in D, the quotient property of convergent sequences implies that

$$\lim_{n\to\infty}\frac{f(x_n)}{g(x_n)} = \frac{f(x_0)}{g(x_0)}. \tag{3.6}$$

By the definition of continuity, (3.1), (3.2), and (3.3) follow from (3.4), (3.5), and (3.6), respectively. ∎

The Polynomial Property for convergent sequences stated in Section 2.1 is precisely the assertion that a polynomial is continuous. Thus, by the quotient property for continuous functions, we have the following corollary describing a general class of continuous functions.

Corollary 3.5 Let $p:\mathbb{R}\to\mathbb{R}$ and $q:\mathbb{R}\to\mathbb{R}$ be polynomials. Then the quotient $p/q: D\to\mathbb{R}$ is continuous, where $D = \{x \text{ in } \mathbb{R} \mid q(x)\neq 0\}$.

Compositions of Continuous Functions

In addition to forming the sum, product, and quotient of functions, there is another useful way to combine functions: They can be *composed.*

Definition For functions $f: D\to\mathbb{R}$ and $g: U\to\mathbb{R}$ such that $f(D)$ is contained in U, we define the composition of $f: D\to\mathbb{R}$ with $g: U\to\mathbb{R}$, denoted by $g\circ f: D\to\mathbb{R}$, by

$$(g\circ f)(x) \equiv g(f(x)) \qquad \text{for all } x \text{ in } D.$$

We have the following composition property for continuous functions.

Theorem 3.6 For functions $f: D\to\mathbb{R}$ and $g: U\to\mathbb{R}$ such that $f(D)$ is contained in U, suppose that $f: D\to\mathbb{R}$ is continuous at the point x_0 in D and $g: U\to\mathbb{R}$ is continuous at the point $f(x_0)$. Then the composition

$$g\circ f: D\to\mathbb{R}$$

Proof

Let $\{x_n\}$ be a sequence in D that converges to x_0. By the continuity of the function $f: D\to\mathbb{R}$ at the point x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. But then $\{f(x_n)\}$ is a sequence in U that converges to $f(x_0)$, so by the continuity of $g: U\to\mathbb{R}$ at the point $f(x_0)$, the sequence $\{g(f(x_n))\}$ converges to $g(f(x_0))$; that is,

$$\lim_{n\to\infty}(g\circ f)(x_n) = (g\circ f)(x_0).$$

Thus, the composition $g\circ f$ is continuous at x_0. ∎

EXERCISES FOR SECTION 3.1

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. If the function $f+g:\mathbb{R}\to\mathbb{R}$ is continuous, then the functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ also are continuous.
   b. If the function $f^2:\mathbb{R}\to\mathbb{R}$ is continuous, then so is the function $f:\mathbb{R}\to\mathbb{R}$.
   c. If the functions $f+g:\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous, then so is the function $f:\mathbb{R}\to\mathbb{R}$.
   d. Every function $f:\mathbb{N}\to\mathbb{R}$ is continuous, where $\mathbb{N}$ denotes the set of natural numbers.
2. Define
$$f(x) = \begin{cases} 11 & \text{if } 0\le x\le 1 \\ x & \text{if } 1 < x \le 2.\end{cases}$$
   At what points is the function $f:[0,2]\to\mathbb{R}$ continuous? Justify your answer.
3. Define
$$f(x) = \begin{cases} x^2 & \text{if } x\le 0 \\ x+1 & \text{if } x > 0.\end{cases}$$
   At what points is the function $f:\mathbb{R}\to\mathbb{R}$ continuous? Justify your answer.
4. For a function $f: D\to\mathbb{R}$ and a point x_0 in D, define $A = \{x \text{ in } D \mid x\ge x_0\}$ and $B = \{x \text{ in } D \mid x \le x_0\}$. Prove that $f: D\to\mathbb{R}$ is continuous at x_0 if and only if $f: A\to\mathbb{R}$ and $f: B\to\mathbb{R}$ are continuous at x_0.
5. Define
$$f(x) = \begin{cases} x^2 & \text{if } x\ge 0 \\ x & \text{if } x < 0.\end{cases}$$
   Prove that the function $f:\mathbb{R}\to\mathbb{R}$ is continuous. (*Hint:* Use Exercise 4.)
6. Define the function $g:\mathbb{R}\to\mathbb{R}$ by
$$g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational.}\end{cases}$$
   At what points is the function continuous? Justify your answer.
7. Suppose that the function $f:[0,1]\to\mathbb{R}$ is continuous and that
$$f(x)\ge 2 \qquad \text{if } 0\le x < 1.$$

8. Suppose that the function $f:[0,1]\to\mathbb{R}$ is continuous and that

$$f(x) > 2 \qquad \text{if } 0 \le x < 1.$$

Is it necessarily the case that $f(1) > 2$?

9. Suppose that the function $f:\mathbb{R}\to\mathbb{R}$ is continuous at the point x_0 and that $f(x_0) > 0$. Prove that there is an interval $I \equiv (x_0-1/n, x_0+1/n)$, where n is a natural number, such that $f(x) > 0$ for all x in I. (*Hint:* Argue by contradiction.)

10. Suppose that the function $f:\mathbb{R}\to\mathbb{R}$ is continuous at the point x_0. Prove that there is an interval $I \equiv (x_0-1/n, x_0+1/n)$, where n is a natural number, such that $f(x) < n$ for all x in I. (*Hint:* Argue by contradiction.)

11. Suppose that the function $g:\mathbb{R}\to\mathbb{R}$ is continuous and that $g(x)=0$ if x is rational. Prove that $g(x) = 0$ for all x in $\mathbb{R}$.

12. Let the function $f: D\to\mathbb{R}$ be continuous. Then define the function $|f|: D\to\mathbb{R}$ by $|f|(x) = |f(x)|$ for x in D. Prove that the function $|f|: D\to\mathbb{R}$ also is continuous.

13. A function $f:D\to\mathbb{R}$ is said to be a *Lipschitz function* provided that there is a nonnegative number C such that

$$|f(u) - f(v)| \le C|u-v| \qquad \text{for all } u \text{ and } v \text{ in } D.$$

Use the Comparison Lemma of Section 2.1 to show that a Lipschitz function is continuous.

14. Suppose that the function $f:\mathbb{R}\to\mathbb{R}$ has the property that

$$f(u+v) = f(u) + f(v) \qquad \text{for all } u \text{ and } v.$$

a. Define $m \equiv f(1)$. Prove that

$$f(x) = mx \qquad \text{for all rational numbers } x.$$

b. Use (a) to prove that if $f:\mathbb{R}\to\mathbb{R}$ is continuous, then

$$f(x) = mx \qquad \text{for all } x.$$

3.2 THE EXTREME VALUE THEOREM

For a function $f: D\to\mathbb{R}$, we define

$$f(D) \equiv \{y \mid y = f(x) \qquad \text{for some } x \text{ in } D\}$$

and call the set $f(D)$ the *image* of the function $f:D\to\mathbb{R}$. We say that the function $f:D\to\mathbb{R}$ attains a *maximum value* provided that its image $f(D)$ has a maximum; that is, there is a point x_0 in D such that

$$f(x) \le f(x_0) \qquad \text{for all } x \text{ in } D.$$

We will call such a point x_0 in D a *maximizer* of the function $f: D\to\mathbb{R}$. Similarly, the function $f: D\to\mathbb{R}$ is said to attain a *minimum value* provided that its image $f(D)$ has a minimum; a point in D at which this minimum value is attained is called a *minimizer* of the function $f: D\to\mathbb{R}$.

In general, a nonempty set has a maximum provided that the set is bounded above and contains its supremum. Thus, a function $f:D\to\mathbb{R}$ has a maximum precisely when the image $f(D)$ is bounded above and the supremum of the image is a functional value.

In general, no assertion can be made concerning the existence of a minimum or maximum value for a function $f: D\to\mathbb{R}$.

Example 3.7 Define the function $f:(0,1)\to\mathbb{R}$ by $f(x) = 2x$ for all x in $(0,1)$. This function does not have a maximum value since no matter what x_0 in $(0,1)$ is chosen, all the points in the interval $(x_0, 1)$ have functional values greater than $f(x_0)$. Observe that the image is bounded above with supremum 2 but that 2 is not attained as a functional value.

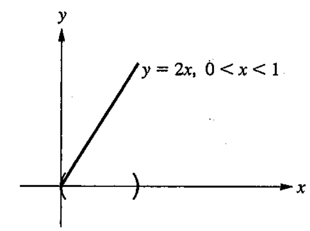

FIGURE 3.2 The supremun of the image is not a functional value. ■

Example 3.8 Define the function $f:(0,1)\to\mathbb{R}$ by $f(x) = 1/x$ for all x in $(0,1)$. For each natural number n, $f(1/n) = n$, so the image is not even bounded above. Thus, the function certainly cannot attain a maximum value.

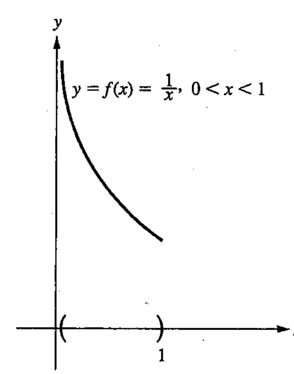

FIGURE 3.3 The image of the function $f(x) = 1/x$ on $(0, 1]$ is not bounded above. ■

However, in the case that the domain D is a closed bounded interval $[a, b]$ and the function $f:[a,b]\to\mathbb{R}$ is continuous, we have the following important result.

c. Prove that if the point x_0 in D is a limit point of D, then a function $f: D \to \mathbb{R}$ is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

9. Suppose the function $f: \mathbb{R} \to \mathbb{R}$ has the property that there is some $M > 0$ such that

$$|f(x)| \leq M|x|^2 \qquad \text{for all } x.$$

Prove that

$$\lim_{x \to 0} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0} \frac{f(x)}{x} = 0.$$

10. For each number x, define $f(x)$ to be the largest integer that is less than or equal to x. Graph the function $f: \mathbb{R} \to \mathbb{R}$. Given a number x_0, examine

$$\lim_{x \to x_0} f(x).$$

11. Let k be a natural number. Prove that

$$\lim_{x \to 1} \frac{x^k - 1}{x - 1} = k.$$

12. (A General Monotone Convergence Principle.) Let a and b be numbers with $a < b$ and set $I = (a, b)$. Suppose that the function $f: I \to \mathbb{R}$ is monotonically increasing and bounded. Prove that $\lim_{x \to a} f(x)$ exists.

CHAPTER 4

DIFFERENTIATION

4.1 THE ALGEBRA OF DERIVATIVES

The simplest type of function $f: \mathbb{R} \to \mathbb{R}$ is one whose graph is a line. For such a function, the ratio

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

where $x_1 \neq x_2$, does not depend on the choice of points x_1 and x_2. We denote this ratio by m and call m the *slope* of the graph of f. So a function f whose graph is a line is completely determined by prescribing its functional value at one point, say at x_0, and then prescribing its slope m; it is then defined by the formula

$$f(x) = f(x_0) + m(x - x_0) \qquad \text{for all } x. \tag{4.1}$$

For a function whose graph is not a line, it makes no sense to speak of "the slope of the graph." However, many functions have the property that at certain points on their graph, the graph can be approximated, in a sense that we will soon make precise, by a tangent line. One then defines the slope of the graph at that point to be the slope of the tangent line. The slope will vary from point to point, and when we can determine the slope at each point we have very useful information for analyzing the function. This is the basic geometric idea behind differentiation.[1]

An open interval $I = (a, b)$ that contains the point x_0 is called a *neighborhood* of x_0.

[1] We will prove a version of formula (4.1) for differentiable functions whose graphs are not lines; it is called the First Fundamental Theorem (Integrating Derivatives): For a differentiable function $f: \mathbb{R} \to \mathbb{R}$ whose derivative is continuous, formula (4.1) becomes

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt \qquad \text{for all } x.$$

The symbols and the formula will be explained in Chapter 6.

Tangent Lines and Derivatives

To make the above precise, we need to define the *tangent line*. For a function $f : I \to \mathbb{R}$, where I is a neighborhood of the point x_0, observe that for a point x in I, with $x \neq x_0$, the slope of the line joining the points $(x_0, f(x_0))$ and $(x, f(x))$ is

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

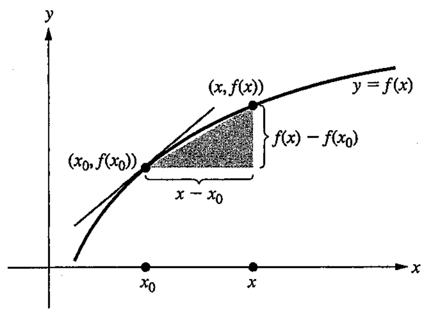

FIGURE 4.1 Approximation of the slope of the tangent line at the point $(x_0, f(x_0))$.

It is reasonable to expect that if there is a tangent line to the graph of $f : I \to \mathbb{R}$ at $(x_0, f(x_0))$, which has a slope m_0, then one should have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_0.$$

For a number x_0, an open interval $I = (a, b)$ that contains x_0 is called a *neighborhood* of x_0.

Definition Let I be a neighborhood of x_0. Then the function $f : I \to \mathbb{R}$ is said to be differentiable at x_0 provided that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{4.2}$$

exists, in which case we denote this limit by $f'(x_0)$ and call it the derivative of f at x_0; that is,

$$f'(x_0) \equiv \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{4.3}$$

If the function $f : I \to \mathbb{R}$ is differentiable at every point in I, we say that f is *differentiable* and call the function $f' : I \to \mathbb{R}$ the *derivative* of f.

For a function $f : I \to \mathbb{R}$ that is differentiable at x_0, we call the line determined by the equation

$$y = f(x_0) + f'(x_0)(x - x_0), \qquad \text{for all } x,$$

the *tangent line to the graph of f at the point* $(x_0, f(x_0))$.

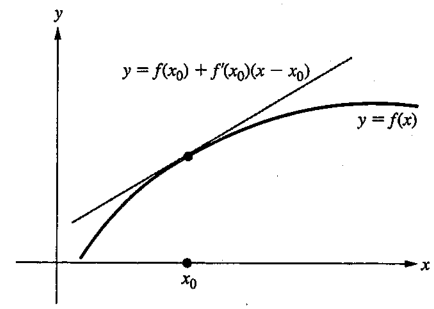

FIGURE 4.2 The tangent line to the graph of f at the point $(x_0, f(x_0))$.

Observe that since $\lim_{x \to x_0}[x - x_0] = 0$, we cannot use the quotient formula for limits in the determination of differentiability. To overcome this obstacle, in this and the next section we will develop techniques for evaluating limits of the type (4.2), which are referred to as *differentiation rules*. Before turning to these, we will consider some specific examples.

Three Examples

Example 4.1 Define $f(x) = mx + b$ for all x. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = m \qquad \text{for all } x.$$

Indeed, for x_0 in $\mathbb{R}$,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{m(x - x_0)}{x - x_0} = m \qquad \text{if } x \neq x_0.$$

Thus,

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} m = m.$$

■

Example 4.2 Consider the simplest polynomial whose graph is not a line. Define $f(x) = x^2$ for all x. Then $f:\mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = 2x \qquad \text{for all } x.$$

Indeed, for x_0 in $\mathbb{R}$, by the difference of squares formula,

$$\frac{f(x)-f(x_0)}{x-x_0} = \frac{x^2-x_0^2}{x-x_0} = \frac{(x-x_0)(x+x_0)}{x-x_0} = x+x_0 \qquad \text{if } x \neq x_0.$$

Thus,

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}[x+x_0] = 2x_0.$$ ■

Example 4.3 Define $f(x) = |x|$ for all x. Then $f:\mathbb{R} \to \mathbb{R}$ is not differentiable at $x = 0$. To see this, observe that

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = 1 \qquad \text{if } x > 0,$$

while

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|}{x} = -1 \qquad \text{if } x < 0.$$

It follows that

$$\lim_{x\to 0, x>0} \frac{f(x)-f(0)}{x-0} = 1 \qquad \text{and} \qquad \lim_{x\to 0, x>0} \frac{f(x)-f(0)}{x-0} = -1.$$

Thus,

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \text{ does not exist.}$$

It is easy to see that if $x \neq 0$, then f is differentiable at x, and $f'(x) = 1$ if $x > 0$, while $f'(x) = -1$ if $x < 0$. ■

Differentiating Positive Integral Powers

Proposition 4.4 For a natural number n, define $f(x) = x^n$ for all x. Then the function $f:\mathbb{R} \to \mathbb{R}$ is differentiable and

$$f'(x) = nx^{n-1} \qquad \text{for all } x.$$

Proof

Fix a number x_0. Observe that by the difference of powers formula,

$$x^n - x_0^n = (x-x_0)(x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-2} + x_0^{n-1}) \qquad \text{for all } x,$$

and hence

$$\frac{f(x)-f(x_0)}{x-x_0} = x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-2} + x_0^{n-1} \qquad \text{if } x \neq x_0.$$

Observe that there are n terms on the right-hand side and that each has x_0^{n-1} as its limit as x approaches x_0. Thus, by the sum property of limits,

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = nx_0^{n-1}.$$ ■

Differentiable Functions are Continuous

Proposition 4.5 Let I be a neighborhood of x_0 and suppose that the function $f: I \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.

Proof

Since

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \qquad \text{and} \qquad \lim_{x\to x_0}[x-x_0] = 0,$$

it follows from the product property of limits that

$$\lim_{x\to x_0}[f(x)-f(x_0)] = \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}\cdot(x-x_0)\right] = f'(x_0)\cdot 0 = 0.$$

Thus, $\lim_{x\to x_0} f(x) = f(x_0)$, which means that f is continuous at x_0. ■

As Example 4.3 shows, it is not true that continuity of a function at a point implies the differentiability of the function at that point.

Differentiating Sums, Products, and Quotients

Theorem 4.6 Let I be a neighborhood of x_0 and suppose that the functions $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ are differentiable at x_0. Then

i. the sum $f + g: I \to \mathbb{R}$ is differentiable at x_0 and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0);$$

ii. the product $fg: I \to \mathbb{R}$ is differentiable at x_0 and

$$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0);$$

iii. if $g(x) \neq 0$ for all x in I, then the reciprocal $1/g: I \to \mathbb{R}$ is differentiable at x_0 and

$$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2};$$

and

iv. if $g(x) \neq 0$ for all x in I, then the quotient $f/g : I \to \mathbb{R}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Proof of (i)
For x in I, with $x \neq x_0$,

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}.$$

Hence, by the definition of derivative and the sum property of limits,

$$\lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = f'(x_0) + g'(x_0).$$ ■

Proof of (ii)
In this proof, observe that in order to facilitate factorization, in the numerator we subtract and add the term $f(x)g(x_0)$. For x in I, with $x \neq x_0$,

$$\begin{aligned}\frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}\\ &= \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}\\ &= f(x)\left[\frac{g(x) - g(x_0)}{x - x_0}\right] + g(x_0)\left[\frac{f(x) - f(x_0)}{x - x_0}\right].\end{aligned}$$

Since differentiability implies continuity, $\lim_{x \to x_0} f(x) = f(x_0)$. Consequently, using the definition of derivative and the sum and product properties for limits,

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = f(x_0)g'(x_0) + g(x_0)f'(x_0).$$ ■

Proof of (iii)
For x in I, with $x \neq x_0$,

$$\begin{aligned}\frac{(1/g)(x) - (1/g)(x_0)}{x - x_0} &= \frac{1/g(x) - 1/g(x_0)}{x - x_0}\\ &= \frac{1}{g(x)g(x_0)}\left[\frac{g(x_0) - g(x)}{x - x_0}\right]\\ &= \frac{-1}{g(x)g(x_0)}\left[\frac{g(x) - g(x_0)}{x - x_0}\right].\end{aligned}$$

Since differentiability implies continuity, $\lim_{x \to x_0} g(x) = g(x_0)$. Hence we can use the definition of derivative, together with the product and quotient properties of limits, to conclude from the preceding identity that

$$\lim_{x \to x_0}\left[\frac{(1/g)(x) - (1/g)(x_0)}{x - x_0}\right] = \frac{-g'(x_0)}{(g(x_0))^2}.$$ ■

Proof of (iv)
For x in I, with $x \neq x_0$, observe that

$$\frac{f(x)}{g(x)} = \frac{1}{g(x)} \cdot f(x).$$

The quotient formula for derivatives now follows from parts (ii) and (iii). ■

Proposition 4.7 For an integer n, define the set $\mathcal{O}$ to be $\mathbb{R}$ if $n \geq 0$ and to be $\{x \text{ in } \mathbb{R} \mid x \neq 0\}$ if $n < 0$. Then define

$$f(x) = x^n \quad \text{for all } x \text{ in } \mathcal{O}.$$

The function $f : \mathcal{O} \to \mathbb{R}$ is differentiable, and

$$f'(x) = nx^{n-1} \quad \text{for all } x \text{ in } \mathcal{O}.$$

Proof
The case in which $n > 0$ is precisely Proposition 4.4, so we need only consider the case $n < 0$. But if $n < 0$, then

$$f(x) = \frac{1}{x^{-n}} \quad \text{for all } x \text{ in } \mathcal{O},$$

where $-n$ is a natural number. Then from Proposition 4.4 and the formula for differentiating the reciprocal of a differentiable function [part (iii) of Theorem 4.6], it follows that $f : \mathcal{O} \to \mathbb{R}$ is differentiable and

$$\begin{aligned}f'(x) &= \frac{-[(-n)x^{-n-1}]}{(x^{-n})^2}\\ &= nx^{n-1} \quad \text{for all } x \text{ in } \mathcal{O}.\end{aligned}$$ ■

Corollary 4.8 For polynomials $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$, define the set $\mathcal{O}$ to be $\{x \text{ in } \mathbb{R} \mid q(x) \neq 0\}$. Then the quotient $p/q : \mathcal{O} \to \mathbb{R}$ is differentiable.

Proof
From Proposition 4.4 and parts (i) and (ii) of Theorem 4.6 it follows that both $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are differentiable. Then part (iv) of Theorem 4.6 implies that $p/q : \mathcal{O} \to \mathbb{R}$ is differentiable. ■

EXERCISES FOR SECTION 4.1

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. If the function $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0, then it is differentiable at x_0.
   b. If the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0.
   c. The function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if the function $f^2 : \mathbb{R} \to \mathbb{R}$ is differentiable.

2. Define $f(x) = x^3 + 2x + 1$ for all x. Find the equation of the tangent line to the graph of $f : \mathbb{R} \to \mathbb{R}$ at the point (2, 13).
3. For m_1 and m_2 numbers, with $m_1 \neq m_2$, define
$$f(x) = \begin{cases} m_1 x + 4 & \text{if } x \leq 0 \\ m_2 x + 4 & \text{if } x \geq 0. \end{cases}$$
Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous but not differentiable at $x = 0$.
4. Use the definition of derivative to compute the derivative of the following functions at $x = 1$:
a. $f(x) = \sqrt{x+1}$ for all $x > 0$.
b. $f(x) = x^3 + 2x$ for all x.
c. $f(x) = 1/(1 + x^2)$ for all x.
5. Evaluate the following limits or determine that they do not exist:
a. $\lim_{x\to 0} \dfrac{x^2}{x}$
b. $\lim_{x\to 1} \dfrac{x^2-1}{\sqrt{x}-1}$
c. $\lim_{x\to 0} \dfrac{x-1}{\sqrt{x}-1}$
d. $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$
6. Let I and J be open intervals, and the functions $f : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ have the property that $h(J) \subseteq I$, so the composition $f \circ h : J \to \mathbb{R}$ is defined. Show that if x_0 is in J, $h : J \to \mathbb{R}$ is continuous at x_0, $h(x) \neq h(x_0)$ if $x \neq x_0$, and $f : I \to \mathbb{R}$ is differentiable at $h(x_0)$, then
$$\lim_{x\to x_0} \frac{f(h(x)) - f(h(x_0))}{h(x) - h(x_0)} = f'(h(x_0)).$$
7. Use Exercise 6 to show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 = 1$, then:
a. $\lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = f'(1)$
b. $\lim_{t\to 1} \dfrac{f(\sqrt{t}) - f(1)}{\sqrt{t}-1} = f'(1)$
c. $\lim_{x\to 1} \dfrac{f(x^2) - f(1)}{x^2-1} = f'(1)$
d. $\lim_{x\to 1} \dfrac{f(x^2) - f(1)}{x-1} = 2f'(1)$
e. $\lim_{x\to 1} \dfrac{f(x^3) - f(1)}{x-1} = 3f'(1)$.
(*Hint:* For the last two limits, first make use of the difference of powers formula.)
8. For a natural number $n \geq 2$, define
$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^n & \text{if } x > 0. \end{cases}$$

9. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has the property that
$$-x^2 \leq f(x) \leq x^2 \qquad \text{for all } x.$$
Prove that f is differentiable at $x = 0$ and that $f'(0) = 0$.
10. For real numbers a and b, define
$$g(x) = \begin{cases} 3x^2 & \text{if } x \leq 1 \\ a + bx & \text{if } x > 1. \end{cases}$$
For what values of a and b is the function $g : \mathbb{R} \to \mathbb{R}$ differentiable at $x = 1$?
11. Suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = 0$. Also, suppose that for each natural number n, $g(1/n) = 0$. Prove that $g(0) = 0$ and $g'(0) = 0$.
12. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and monotonically increasing. Show that $f'(x) \geq 0$ for all x.
13. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a bounded sequence $\{x_n\}$ with $x_n \neq x_m$, if $n \neq m$, such that $f(x_n) = 0$ for every index n. Show that there is a point x_0 at which $f(x_0) = 0$ and $f'(x_0) = 0$. (*Hint:* Use the Sequential Compactness Theorem.)
14. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Analyze the limit
$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{h}.$$
[*Hint:* Subtract and add $f(x_0)$ to the numerator.]
15. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0. Prove that
$$\lim_{x\to x_0} \frac{xf(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0).$$
16. Let the function $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 0$. Prove that
$$\lim_{x\to 0} \frac{f(x^2) - f(0)}{x} = 0.$$
17. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 0. For real numbers a, b, and c, with $c \neq 0$, prove that
$$\lim_{x\to 0} \frac{f(ax) - f(bx)}{cx} = \left[\frac{a-b}{c}\right] f'(0).$$
18. Let the function $h : \mathbb{R} \to \mathbb{R}$ be bounded. Define the function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = 1 + 4x + x^2 h(x) \qquad \text{for all } x.$$
Prove that $f(0) = 1$ and $f'(0) = 4$. (*Note:* There is no assumption about the

any positive irrational number z; for instance, choose $z = \sqrt{2}$. By the density of the rationals there is a rational number x in the interval $(a/z, b/z)$ so that zx lies in the interval (a, b) and zx is irrational since it is the product of an irrational number and a rational number. ■

EXERCISES FOR SECTION 1.2

1. For each of the following statements, determine whether it is true or false and justify your answer.
   a. The set $\mathbb{Z}$ of integers is dense in $\mathbb{R}$.
   b. The set of positive real numbers is dense in $\mathbb{R}$.
   c. The set $\mathbb{Q} \setminus \mathbb{N}$ of rational numbers that are not integers is dense in $\mathbb{R}$.
2. Suppose that S is a nonempty set of integers that is bounded below. Show that S has a minimum. In particular, conclude that every nonempty set of natural numbers has a minimum.
3. Let S be a nonempty set of real numbers that is bounded below. Prove that the set S has a minimum if and only if the number $\inf S$ belongs to S.
4. For each of the following two sets, find the maximum, minimum, infimum, and supremum if they are defined. Justify your conclusions.
   a. $\{1/n \mid n \text{ in } \mathbb{N}\}$
   b. $\{x \text{ in } \mathbb{R} \mid x^2 < 2\}$
5. Suppose that the number a has the property that for every natural number n, $a \leq 1/n$. Prove that $a \leq 0$.
6. Given a real number a, define $S \equiv \{x \mid x \text{ in } \mathbb{Q}, x < a\}$. Prove that $a = \sup S$.
7. Show that for any real number c, there is exactly one integer in the interval $(c, c+1]$.
8. Show that the Archimedean Property is a consequence of the assertion that for any real number c, there is an integer in the interval $[c, c+1)$.
9. Show that the Archimedean Property is a consequence of the assertion that every interval (a, b) contains a rational number.

1.3 INEQUALITIES AND IDENTITIES

Recall that for a real number x, its *absolute value,* denoted by $|x|$, is defined by

$$|x| \equiv \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Directly from this definition and from the Positivity Axioms for $\mathbb{R}$, it follows that if c and d are any numbers such that $d \geq 0$, then

$$|c| \leq d \quad \text{if and only if } -d \leq c \leq d. \tag{1.6}$$

Moreover, we also have, for any number x,

$$-|x| \leq x \leq |x|. \tag{1.7}$$

Given a pair of real numbers a and b, we often need to estimate the size of $|a+b|$. The following inequality is a basic tool.

Theorem 1.11 The Triangle Inequality For any pair of numbers a and b,

$$|a+b| \leq |a| + |b|.$$

Proof

Using (1.6), we see that the Triangle Inequality is equivalent to the assertion that

$$-|a| - |b| \leq a + b \leq |a| + |b|. \tag{1.8}$$

However, setting $x = a$ and then $x = b$ in (1.7), we have

$$-|a| \leq a \leq |a| \quad \text{and} \quad -|b| \leq b \leq |b|$$

from which, by addition, we obtain (1.8) and hence the Triangle Inequality. ■

It is useful to explicitly record the following proposition.

Proposition 1.12 For a number a and a positive number r, the following three assertions about a number x are equivalent:

i. $|x - a| < r$.
ii. $a - r < x < a + r$.
iii. x belongs to the open interval $(a - r, a + r)$.

Proof

The equivalence of (i) and (ii) follows from (1.6), while the equivalence of (ii) and (iii) is simply the very definition of the interval $(a - r, a + r)$. ■

At the heart of many arguments in analysis lies the problem of estimating the sizes of various quotients, differences, and sums and of simplifying various algebraic expressions. As a companion tool to the Triangle Inequality we now establish three useful algebraic identities.

For a natural number n and any number a, as usual, we write a^n to denote the product of a multiplied by itself n times.

Observe that we have the following formulas for the difference of squares and the difference of cubes:

$$a^2 - b^2 = (a-b)(a+b) \qquad \text{and} \qquad a^3 - b^3 = (a-b)(a^2+ab+b^2).$$

These are special cases of the following formula.

The Difference of Powers Formula

For any natural number n and any numbers a and b,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

It is easy to verify this formula just by expanding the right-hand side. Indeed,

$$\begin{aligned} &(a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \\ &= a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + a^2b^{n-2} + ab^{n-1} \\ &\quad - a^{n-1}b - a^{n-2}b^2 - \cdots - a^2b^{n-2} - ab^{n-1} - b^n \\ &= a^n - b^n. \end{aligned}$$

In the Difference of Powers Formula, if we take $a \equiv 1$, set $b \equiv r \neq 1$, and replace n by $n+1$, then after division by $1 - T$ we obtain the following important identity.

The Geometric Sum Formula

For any natural number n and any number $r \neq 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

This formula is the essential tool underlying the frequent possibility of expressing functions as power series that we will consider in Chapters 8 and 9. It also plays an essential role in verifying many computational algorithms.

It will be useful to have a formula that expresses powers of the sum of the numbers a and b in terms of the powers of a and of b. In order to state this formula, we need to introduce factorial notation. For each natural number n, we define the symbol $n!$, which is called *n factorial*, as follows: We define $1! \equiv 1$, and if k is any natural number for which $k!$ has been defined, we then define $(k+1)! \equiv (k+1)k!$. By the Principle of Mathematical Induction, the symbol $n!$ is defined for all natural numbers n. It is convenient to define $0! \equiv 1$. We also need to introduce, for each pair of nonnegative integers n and k such that $n \geq k$, the *binomial coefficient* $\binom{n}{k}$, which is defined by the formula

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}.$$

We have the following formula for $(a+b)^n$, a proof of which is outlined in Exercises 21 and 22.

The Binomial Formula

For each natural number n and each pair of numbers a and b,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

We close this discussion on algebraic identities by recalling the summation notation. For a natural number n and numbers $a_0, a_1, \ldots, a_n$, we define

$$\sum_{k=0}^{n} a_k \equiv a_0 + a_1 + \cdots + a_n.$$

This notation shortens many formulas. For instance, using this summation notation, the three algebraic formulas we have described become the following.

The Difference of Powers Formula

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k.$$

The Geometric Sum Formula

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \qquad \text{if } r \neq 1.$$

The Binomial Formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.$$

EXERCISES FOR SECTION 1.3

1. Write out the Difference of Powers Formula explicitly for $n = 4$ and 5.
2. Write out the Binomial Formula explicitly for $n = 2$, 3, and 4.
3. Show that the Triangle Inequality becomes an equality if a and b are of the same sign.
4. Let $a > 0$. Prove that if x is a number such that $|x - a| < a/2$, then $x > a/2$.
5. Let $b < 0$. Prove that if x is a number such that $|x - b| < |b|/2$, then $x < b/2$.
6. Which of the following inequalities hold for all numbers a and b? Justify your conclusions.
   a. $|a + b| \geq |a| + |b|$.
   b. $|a + b| \leq |a| - |b|$.
7. By writing $a = (a + b) + (-b)$ use the Triangle Inequality to obtain $|a| - |b| \leq |a + b|$. Then interchange a and b to show that

$$||a| - |b|| < |a + b|.$$

Then replace b by $-b$ to obtain

$$||a| - |b|| \le |a - b|.$$

8. Let a and b be numbers such that $|a - b| \le 1$. Prove that $|a| \le |b| + 1$.

9. For a natural number n and any two nonnegative numbers a and b, use the Difference of Powers Formula to prove that

$$a \le b \qquad \text{if and only if } a^n \le b^n.$$

10. For a natural number n and numbers a and b such that $a \ge b \ge 0$, prove that

$$a^n - b^n \ge nb^{n-1}(a - b).$$

11. (Bernoulli's Inequality) Show that for a natural number n and a nonnegative number b,

$$(1 + b)^n \ge 1 + nb.$$

(*Hint:* In the Binomial Formula, set $a = 1$.)

12. Use the Principle of Mathematical Induction to provide a direct proof of Bernoulli's Inequality for all $b > -1$, not just for the case where $b \ge 0$ which, as outlined in Exercise 11 follows from the Binomial Formula.

13. For a natural number n and a nonnegative number b show that

$$(1 + b)^n \ge 1 + nb + \frac{n(n-1)}{2}b^2.$$

14. (Cauchy's Inequality) Using the fact that the square of a real number is nonnegative, prove that for any numbers a and b,

$$ab \le \frac{1}{2}(a^2 + b^2).$$

15. Use Cauchy's Inequality to prove that if $a \ge 0$ and $b \ge 0$, then

$$\sqrt{ab} \le \frac{1}{2}(a + b).$$

16. Use Cauchy's Inequality to show that for any numbers a and b and a natural number n,

$$ab \le \frac{1}{2}\left(na^2 + \frac{1}{n}b^2\right).$$

(*Hint:* Replace a by $\sqrt{n}a$ and b by $b/\sqrt{n}$ in Cauchy's Inequality.)

17. Let a, b, and c be nonnegative numbers. Prove the following inequalities:

a. $ab + bc + ca \le a^2 + b^2 + c^2$.

b. $8abc \le (a + b)(b + c)(c + a)$.

c. $abc(a + b + c) \le a^2b^2 + b^2c^2 + c^2a^2$.

18. A function $f : \mathbb{R} \to \mathbb{R}$ is called *strictly increasing* provided that $f(u) > f(v)$ for all numbers u and v such that $u > v$.

a. Define $p(x) = x^3$ for all x. Prove that the polynomial $p : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

b. Fix a number c and define $q(x) = x^3 + cx$ for all x. Prove that the polynomial $q : \mathbb{R} \to \mathbb{R}$ is strictly increasing if and only if $c \ge 0$. (*Hint:* For $c < 0$, consider the graph to understand why it is not strictly increasing and then prove it is not increasing.)

19. Let n be a natural number and $a_1, a_2, \ldots, a_n$ be positive numbers. Prove that

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) \ge 1 + a_1 + a_2 + \cdots + a_n$$

and that

$$(a_1 + a_2 + \cdots + a_n)\left(a_1^{-1} + a_2^{-1} + \cdots + a_n^{-1}\right) \ge n^2.$$

20. Use the Geometric Sum Formula to find a formula for

a. $\dfrac{1}{1 + x^2} + \dfrac{1}{(1 + x^2)^2} + \cdots + \dfrac{1}{(1 + x^2)^n}$.

b. Also, show that if $a \ne 0$, then

$$\frac{1}{a} = 1 + (1 - a) + (1 - a)^2 + \frac{(1 - a)^3}{a}.$$

21. Prove that if n and k are natural numbers such that $k \le n$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

22. Use the formula in Exercise 21 to provide an inductive proof of the Binomial Formula.

23. Let a be a nonzero number and m and n be integers. Prove the following equalities:

a. $a^{m+n} = a^m a^n$.

b. $(ab)^n = a^n b^n$.

24. A natural number n is called *even* if it can be written as $n = 2k$ for some other natural number k, and is called *odd* if either $n = 1$ or $n = 2k + 1$ for some other natural number k.

a. Prove that each natural number n is either odd or even.

b. Prove that if m is a natural number, then $2m > 1$.

c. Prove that a natural number n cannot be both odd and even. (*Hint:* Use part (b).)

d. Suppose that k_1, k_2, ℓ_1, and ℓ_2 are natural numbers such that ℓ_1 and ℓ_2 are odd. Prove that if $2^{k_1}\ell_1 = 2^{k_2}\ell_2$, then $k_1 = k_2$ and $\ell_1 = \ell_2$.

25. **a.** Prove that if n is a natural number, then $2^n > n$.

b. Prove that if n is a natural number, then

$$n = 2^{k_0}\ell_0$$