**MATH 310: Groupwork 11 Solutions** **Tuesday 7/24/2012**

1. For each of the following indicate symoblically what you would assume for each proof method: Direct, by contrapositive and by contradiction.
   (a) $\forall x, (P(x) \wedge Q(x)) \rightarrow (R(x) \vee S(x))$
   Direct: $(P(x) \wedge Q(x))$
   Contrapositive: $((\sim R(x)) \wedge (\sim S(x)))$
   Contradiction: $\exists x, (P(x) \wedge Q(x)) \wedge ((\sim R(x)) \wedge (\sim S(x)))$
   (b) $\forall x, \exists y, P(x,y) \rightarrow (Q(x) \wedge R(y))$
   Direct: $P(x,y)$
   Contrapositive: $((\sim Q(x)) \vee (\sim R(y)))$
   Contradiction: $\exists x, \forall y, P(x,y) \wedge ((\sim Q(x)) \vee (\sim R(y)))$

2. Prove by contradiction that $\sqrt{2}+\sqrt{3}$ is irrational.

   Proof: Assume that $\sqrt{2}+\sqrt{3}$ is rational so that $\sqrt{2}+\sqrt{3} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. Then

$$\begin{aligned}
\sqrt{2}+\sqrt{3} &= \frac{a}{b} \\
(\sqrt{2}+\sqrt{3})^2 &= \left(\frac{a}{b}\right)^2 \\
2+2\sqrt{6}+3 &= \frac{a^2}{b^2} \\
2\sqrt{6} &= \frac{a^2}{b^2} - 5 \\
\sqrt{6} &= \frac{a^2}{2b^2} - \frac{5}{2}
\end{aligned}$$

   which contradicts the lemma that $\sqrt{6}$ is irrational.

3. Prove that $\exists x, y \in \mathbb{Z}, (10x - 8y = 4)$ has a solution.

   Proof: One example is $x = 2$, $y = 2$.

4. Prove that there are infinitely many solutions to the above equation.

   Proof: We first see a pattern of solutions:
   $x = 2$, $y = 2$
   $x = 6$, $y = 7$
   $x = 10$, $y = 12$ So we conjecture that $x = 2 + 4t$ and $y = 2 + 5t$ for $t \in \mathbb{Z}$. We check:

$$10(2+4t) - 8(2+5t) = 20 + 40t - 16 - 20t = 4$$

   and it works.

5. Prove that the equation $x^3 - 9x + 5 = 0$ has two real solutions.

   Proof: Observe that at:
   $x = -4$: $(-4)^3 - 9(-4) + 5 = -23$
   $x = 0$: $0^3 - 9(0) + 5 = 5$
   $x = 1$: $1^3 - 9(1) + 5 = -3$
   Since $x^3 - 9x + 5$ is continuous the Intermediate Value Theorem guarantees a zero in the interval $(-4, 0)$ and another in $(0, 1)$.