1. Define the recursive sequence

$$a_1 = 1,\ a_2 = 3 \text{ and } a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Use strong induction to prove that $a_n = 2n - 1$ for all $n \in \mathbb{N}$.

Proof: We have:
Base Cases: We check $a_1 = 1 = 2(1) - 1$ which is true and $a_2 = 3 = 2(2) - 1$ which is true.
Inductive Step: We assume $a_i = 2i - 1$ for $1 \leq i \leq n$ and we show $a_{n+1} = 2(n+1) - 1$. Observe that

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} \\ &= 2(2n-1) - (2(n-1) - 1) \\ &= 4n - 2 - 2n + 3 \\ &= 2n + 1 \\ &= 2(n+1) - 1 \end{aligned}$$

$\mathcal{QED}$

Note: In the inductive step we refer back to $n - 1$ so we need $n - 1 \geq 1$ and so $n \geq 2$. This is why we check bases cases up to 2.

2. Prove that any postage greater than or equal to 12 cents can be made from 4 and 5 cent stamps.

Proof: We have:
Base Cases: There are four of them,
$n = 12 = 4 + 4 + 4$
$n = 13 = 5 + 4 + 4$
$n = 14 = 5 + 5 + 4$
$n = 15 = 5 + 5 + 5$
Inductive Step: We assume that i cents can be made for $12 \leq i \leq n$ and we show that $n + 1$ cents can be made. Observe that $n + 1 = (n - 3) + 4$ so we simply make $n - 3$ cents and then add on a 4 cent stamp. $\mathcal{QED}$

Note: In the inductive step we refer back to $n - 3$ so we need $n - 3 \geq 12$ and so $n \geq 15$. This is why we check base cases up to 15.