MATH 310: Groupwork 16 Solutions **Tuesday 7/31/2012**

1. Each of the following is not an equivalence relation because it fails for exactly one criterion. Determine which one and provide evidence. You do not need to prove that the other criteria are satisfied.

   (a) The relation R on $\mathbb{Z}$ defined by aRb iff $a|b$.

   Solution: This fails symmetry. Counterexample: $2R4$ but $4\not R\,2$.

   (b) The relation R on $\mathbb{Z}$ defined by aRb iff both are even.

   Solution: This fails reflexivity. Counterexample: $1\not R\,1$.

   (c) The relation R on $\{2, 3, 4, 5, ...\}$ defined by aRb iff a and b share a common factor greater than 1.

   Solution: This fails transitivity. Counterexample: $4R12$ and $12R3$ but $4\not R\,3$.

2. Define a relation R on $\mathbb{Z}$ by $R = \{(x, y) \mid 4|(3x-7y)\}$. Show that R is an equivalence relation and list the equivalence classes of R.

   Proof: We need to show:

   - Reflexivity: For any $x \in \mathbb{Z}$ we see that $4|(3x - 7x)$ and so xRx.
   - Symmetry: The symmetry turned out to be harder than expected since if we follow the method we did in class we get ugly fractions.
     Assume xRy so that $4m = 3x - 7y$ for $m \in \mathbb{Z}$. Then observe that $3y - 7x = 3(3x - 7y) + (-16x + 24y) = 3(4m) + 4(-4x + 6y) = 4(3m - 4x + 6y)$ so that yRx.
   - Transitivity: Assume xRy and yRz so that $4n = 3x - 7y$ and $4m = 3y - 4z$ for $n, m \in \mathbb{Z}$. Then observe that $3x - 7y = (4m + 7y) - (3y - 4n) = 4(m + y + n)$ so that xRz.

   The equivalence classes are calculated as follow:
   $[0] = \{x \mid xR0\}$ and $xR0$ iff $4|3x$ and so $[0] = \{..., -8, -4, 0, 4, 8, ...\}$.
   $[1] = \{x \mid xR1\}$ and $xR1$ iff $4|3x - 7$ and so $[1] = \{..., -7, -3, 1, 5, 9, ...\}$.
   $[2] = \{x \mid xR2\}$ and $xR2$ iff $4|3x - 14$ and so $[2] = \{..., -6, -2, 2, 6, 10, ...\}$.
   $[3] = \{x \mid xR3\}$ and $xR3$ iff $4|3x - 21$ and so $[3] = \{..., -5, -1, 3, 7, 11, ...\}$.