MATH 310: Groupwork 18 Solutions **Thursday 8/2/2012**

1. Prove that the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{x}{x+1}$ is injective.

   Proof: Let $x_1, x_2 \in [0, \infty)$ with $f(x_1) = f(x_2)$. Then

$$\frac{x_1}{x_1 + 1} = \frac{x_2}{x_2 + 1}$$
$$x_1 x_2 + x_1 = x_1 x_2 + x_2$$
$$x_1 = x_2$$

   $\mathcal{QED}$

2. Prove that the function $f : \mathbb{Z} \to \left\{ \left(\frac{1}{2}\right)^a \mid a \in \mathbb{Z}, a \geq 0 \right\}$ given by $f(x) = \left(\frac{1}{2}\right)^{|2-x|}$ is surjective and sketch the range on the number line.

   Proof: Let $y \in \left\{ \left(\frac{1}{2}\right)^a \mid a \in \mathbb{Z}, a \geq 0 \right\}$. We need to find an $x \in \mathbb{Z}$ with $f(x) = y$. Well $y = \left(\frac{1}{2}\right)^a$ for some $a \in \mathbb{Z}$ with $a \geq 0$ and so we need:

$$f(x) = \left(\frac{1}{2}\right)^a$$
$$\left(\frac{1}{2}\right)^{|2-x|} = \left(\frac{1}{2}\right)^a$$
$$|2 - x| = a$$
$$2 - x = \pm a$$
$$x = 2 \pm a$$

   Note: If it bothers you that we've found x in terms of a rather than in terms of y then just mention the fact that $a = \log_{1/2} y$ and so $x = 2 \pm \log_{1/2} y$. In any case we've found x for any y and proven surjectivity.

   $\mathcal{QED}$

3. Prove that the function $f : (\mathbb{R} - \{0, 1\}) \to (\mathbb{R} - \{0\})$ given by $f(x) = \frac{1}{x(x-1)}$ is surjective.

   Note: See note on groupwork regarding error before proceeding.

   Let $y \in \mathbb{R} - \{0\}$. We wish to find an $x \in \mathbb{R} - \{0, 1\}$ with $f(x) = y$. This means we need:

$$f(x) = y$$
$$\frac{1}{x(x-1)} = y$$
$$1 = yx^2 - yx$$
$$yx^2 - yx - 1 = 0$$
$$x = \frac{y \pm \sqrt{y^2 + 4y}}{2y}$$

   Thus x exists, at least for $y \in \mathbb{R} - (-4, 0]$. $\mathcal{QED}$