1. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 2x^2 - x - 1$ has no inverse. Hint: Go root-hunting!
Proof: Observe that:

$$\begin{aligned} f(-3) &= -27 + 18 + 3 - 1 = -7 < 0 \\ f(-1) &= -1 + 2 + 1 - 1 = 1 > 0 \\ f(0) &= -1 < 0 \end{aligned}$$

and so by the Intermediate Value Theorem there is some $x_1 \in (-3, 1)$ and some $x_2 \in (-1, 0)$ with $f(x_1) = f(x_2) = 0$. Thus f is not injective and has no inverse.

QED

2. Define $f : (\mathbb{R} - \{2\}) \to \mathbb{R}$ by $f(x) = \frac{x}{x-2}$. Find the range of f and then find f^{-1} on that set.
Solution: The range is the set of all $y \in \mathbb{R}$ such that we can find an x with $f(x) = y$. That means we need all y for which $\frac{x}{x-2} = y$ has a solution:

$$\begin{aligned} \frac{x}{x-2} &= y \\ x &= xy - 2y \\ x - xy &= -2y \\ xy - x &= 2y \\ x(y-1) &= 2y \\ x &= \frac{2y}{y-1} \end{aligned}$$

This guarantees a value of x for any $y \in \mathbb{R} - \{1\}$. Notice that we should check that we never get $x = 2$ because it's not in the domain. But if we did then we'd have

$$\begin{aligned} 2 &= \frac{2y}{y-1} \\ 2y - 2 &= 2y \\ -2 &= 0 \end{aligned}$$

which is a contradiction.

To find the inverse we need to solve $f(f^{-1}(y)) = y$ for $f^{-1}(y)$:

$$\begin{aligned} f(f^{-1}(y)) &= y \\ \frac{f^{-1}(y)}{f^{-1}(y) - 2} &= y \\ f^{-1}(y) &= \frac{2y}{y-1} \end{aligned}$$

3. Define $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ by $f([x]) = [5x + 2]$. Show that f has an inverse by showing that it is injective and surjective.

Proof: Observe that:

$$\begin{aligned} f([0]) &= [5(0) + 2] = [2] \\ f([1]) &= [5(1) + 2] = [1] \\ f([2]) &= [5(2) + 2] = [0] \\ f([3]) &= [5(3) + 2] = [5] \\ f([4]) &= [5(4) + 2] = [4] \\ f([5]) &= [5(5) + 2] = [3] \end{aligned}$$

From which we see both surjectivity and injectivity easily.

QED

Note: If the sets are finite and small it's easy to just see what the individual elements do.