MATH 310: Groupwork 21 Solutions **Wednesday 8/8/2012**

1. Disprove by counterexample:

   (a) $|A| < |B|$ and A countable implies B countable.

   Solution: One counterexample is $A = \mathbb{Z}$ and $B = \mathbb{R}$.

   (b) $|A| < |B|$ and B uncountable implies A uncountable.

   Solution: One counterexample is $A = \mathbb{Z}$ and $B = \mathbb{R}$.

2. For each of the following sets write down a few elements which give a good idea of what sorts of elements each set has. Be creatively interesting.

   (a) $\mathbb{Z}$

   Solution: 4,5,2,-1,...

   (b) $\mathcal{P}(\mathbb{Z})$

   Solution: $\emptyset$, $\mathbb{Z}$, $\{1, 2, 3\}$, $\{\text{evens}\}$, $\{5, 6, 7, ...\}$

   (c) $\mathcal{P}(\mathcal{P}(\mathbb{Z}))$

   Solution: $\emptyset$, $\{\mathbb{Z}, \{1, 2\}$, $\{\{1\}, \{\text{evens}\}$

   (d) $\mathcal{P}(\mathbb{R})$.

   Solution: $\emptyset$, $\mathbb{R}$, $\{\pi\}$, $\mathbb{Q}$, $\{1, 1.1\}$

3. Prove that for $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ we have $\Big|[a, b]\Big| = \Big|[c, d]\Big|$ by finding an explicit bijection between the sets.

   Proof: The easiest bijection is the straight line connecting (a, c) with (b, d). This has slope $\frac{d-c}{b-a}$ and point (a, c) and hence has equation $y - c = \left(\frac{d-c}{b-a}\right)(x - a)$ or

   $$f(x) = \left(\frac{d-c}{b-a}\right)(x - a) + c$$

   $\mathcal{QED}$

4. Your hotel now has uncountably infinitely many rooms each numbered with a real number in $[0, 1]$ and all full. However uncountably infinitely many guests arrive, each with a number in $[0, 1]$. Explain how you can fit them all in.

   Solution: First move the guest in room x to room $x/3$. At this point rooms $[0, 1/3]$ are full and $(1/3, 1]$ are empty. Then move new guest x to room $0.5x + 0.5$.