MATH 310: Groupwork 24 Solutions **Monday 8/13/2012**

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 3x + 3$. Prove that $f(x)$ is continuous at $x = 3$.

   Idea: Assume that $\{x_n\}$ converges to 3. We claim $\{f(x_n)\} = \{x_n^2 + 3x_n + 3\}$ converges to $f(3) = 21$. This means that for any ϵ we need an N so that $n > N$ implies $|x_n^2+3x_n+3-21| < \epsilon$. But this inequality is $|x_n - 3||x_n + 6| < \epsilon$. We know we can make $|x_n - 3|$ as small as we want but we need to deal with $|x_n + 6|$. To do this notice that we can find some N so that for $n > N$ we have

   $$\begin{gathered} |x_n - 3| < 1 \\ -1 < x_n - 3 < 1 \\ 8 < x_n + 6 < 10 \\ |x_n + 6| < 10 \end{gathered}$$

   For this N then we'd have $|x_n - 3||x_n + 6| < 10|x_n - 3|$ and just need $10|x_n - 3| < \epsilon$.

   Proof: Assume that $\{x_n\}$ converges to 3. Choose N_1 so that $n > N_1$ implies $|x_n - 3| < 1$ from whence comes $|x_n + 6| < 10$. Choose N_2 so that $n > N_2$ implies $|x_n - 3| < \epsilon/10$. Let $N = \max\{N_1, N_2\}$ so that $n > N$ implies

   $$|x_n^2 + 3x_n + 3 - 21| = |x_n - 3||x_n + 6| < 10|x_n - 3| < 10(\epsilon/10) = \epsilon$$

   as desired.

   QED

2. Define $f : \mathbb{R} \to \mathbb{R}$ by

   $$f(x) = \begin{cases} x - 3 & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$$

   Prove that f is not continuous at $x = 2$.

   Proof: Consider that $\{2 - 1/n\}$ converges to 2 but $\{f(2 - 1/n)\} = \{(2 - 1/n) - 3\}$ converges to $-1 \neq f(2)$.

   QED

3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x^2 - x + 1$. Show that $f'(3) = 11$.

   Proof: Suppose $\{x_n\}$ converges to 3. Then

   $$\left\{\frac{f(x_n) - f(3)}{x_n - 3}\right\} = \left\{\frac{2x_n^2 - x^n + 1 - 16}{x_n - 3}\right\} = \left\{\frac{(x_n - 3)(2x_n + 5)}{x_n - 3}\right\} = \{2x_n + 5\}$$

   which converges to $2(3) + 5 = 11$.

   QED