1. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} -2x+8 & \text{if } x < 3 \\ x-1 & \text{if } x \geq 3 \end{cases}$$
Show that $f'(3)$ is undefined.

   Proof: Assume by way of contradiction that $f'(3) = a$ with $a \in \mathbb{R}$.
   Then if $\{x_n\} = \{3 - 1/n\}$ we have
$$\left\{\frac{f(x_n) - f(3)}{x_n - 3}\right\} = \left\{\frac{-2(3-1/n)+8-2}{3-1/n-3}\right\} = \left\{\frac{2/n}{-1/n}\right\} = \{-2\}$$
which converges to -2 so $a = -2$.
   And if $\{x_n\} = \{3 + 1/n\}$ we have
$$\left\{\frac{f(x_n) - f(3)}{x_n - 3}\right\} = \left\{\frac{3+1/n-1-2}{3+1/n-3}\right\} = \left\{\frac{1/n}{1/n}\right\} = \{1\}$$
which converges to 1 so $a = 1$. Since $-2 \neq 1$ we have a contradiction.

   QED

2. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ -x^2 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$
Show that $f'(0) = 0$.

   Proof (excluding special cases): Suppose $\{x_n\}$ converges to 0. We divide $\{x_n\}$ into two sequences, $\{y_n\}$ consisting of all rationals and $\{z_n\}$ consisting of all irrationals. Both converge to 0 too.
   Then
$$\left\{\frac{f(y_n) - f(0)}{y_n - 0}\right\} = \left\{\frac{y_n^2}{y_n}\right\} = \{y_n\}$$
which converges to 0
   And
$$\left\{\frac{f(z_n) - f(0)}{z_n - 0}\right\} = \left\{\frac{-z_n}{z_n}\right\} = \{-z_n\}$$
which converges to 0.
   Therefore $\{x_n\}$ converges to 0.

   QED

3. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

Show that f is continuous at $x = 1$ and $f'(1) = 2$.

Proof (excluding special cases): Suppose $\{x_n\}$ converges to 1. We divide $\{x_n\}$ into two sequences, $\{y_n\}$ all in $(-\infty, 1]$ and $\{z_n\}$ all in $(1, \infty)$. Both converge to 1 too.
Continuity: Observe that $\{f(y_n)\} = \{y_n^2\}$ converges to 1 and $\{f(z_n)\} = \{2z_n - 1\}$ converges to 1 and so $\{f(x_n)\}$ converges to 1 and $1 = f(1)$.
Derivative: Observe that

$$\left\{ \frac{f(y_n) - f(1)}{y_n - 1} \right\} = \left\{ \frac{y_n^2 - 1}{y_n - 1} \right\} = \{y_n + 1\}$$

converges to 2 and

$$\left\{ \frac{f(z_n) - f(1)}{z_n - 1} \right\} = \left\{ \frac{2z_n - 1 - 1}{z_n - 1} \right\} = \{2\}$$

converges to 2 and so $\left\{ \frac{f(x_n) - f(1)}{x_n - 1} \right\}$ converges to 2.

$\mathcal{QED}$