1. Let $a \in \mathbb{Z}$. Show that if a divided by 5 leaves a remainder of 2 then $5|(a^2-4)$.
   Proof: We have $a = 5k+2$ for $k \in \mathbb{Z}$ and so $a^2 - 4 = (5k+2)^2 - 4 = 25k^2 + 20k = 5(5k^2+4k) = 5b$ for $b = 5k^2 + 4k \in \mathbb{Z}$ and so $5|(a^2-4)$.

2. Let $m \in \mathbb{Z}$. Prove if $3 \nmid m$ then $3 \nmid m^2$.
   Proof: If $3 \nmid m$ then either $m = 3n+1$ or $m = 3n+2$ with $n \in \mathbb{Z}$.
   Case 1: If $m = 3n+1$ then $m^2 = 9m^2 + 6m + 1 = 3(3m^2+2m)+1$ so $3 \nmid m^2$.
   Case 2: If $m = 3n+2$ then $m^2 = 9m^2 + 12m + 4 = 3(3m^2+4m+1)+1$ so $3 \nmid m^2$.

3. Prove or provide a counterexample: Let $a, b, c \in \mathbb{Z}$. If $a|bc$ then either $a|b$ or $a|c$.
   Proof: False. For example if $a = 6$, $b = 2$ and $c = 3$ then $a|bc$ but $a \nmid b$ and $a \nmid c$.

4. Let $a \in \mathbb{Z}$. Prove that $a^3 \equiv a \pmod 3$.
   Proof: We have either $a \equiv 0, 1, 2 \pmod 3$.
   Case 1: If $a \equiv 0 \pmod 3$ then $a = 3k$ for $k \in \mathbb{Z}$ and so $a^3 - a = 27k^3 - 3k = 3(9k^3 - k)$ and so $a^3 \equiv a \pmod 3$.
   Case 2: If $a \equiv 1 \pmod 3$ then $a = 3k+1$ for $k \in \mathbb{Z}$ and so $a^3 - a = 27k^3 + 27k^2 + 6k = 3(9k^3 + 9k^2 + 2k)$ and so $a^3 \equiv a \pmod 3$.
   Case 3: If $a \equiv 2 \pmod 3$ then $a = 3k+2$ for $k \in \mathbb{Z}$ and so $a^3 - a = 27k^3 + 54k^2 + 33k + 6 = 3(9k^3 + 18k^2 + 11k + 2)$ and so $a^3 \equiv a \pmod 3$.
   We see in all cases that $a^3 \equiv a \pmod 3$.

5. Show that if $a \in \mathbb{Z}$ then $a^2 \not\equiv 2 \pmod 4$ and $a^2 \not\equiv 3 \pmod 4$.
   Proof: We have either $a \equiv 0, 1, 2, 3 \pmod 4$.
   Case 1: If $a \equiv 0 \pmod 4$ then $a = 4k$ for $k \in \mathbb{Z}$ and so $a^2 = 16k^2 = 4(4k^2)$ and so $a^2 \equiv 0 \pmod 4$.
   Case 2: If $a \equiv 1 \pmod 4$ then $a = 4k+1$ for $k \in \mathbb{Z}$ and so $a^2 = 16k^2 + 8k + 1 = 4(4k^2+2k)+1$ and so $a^2 \equiv 1 \pmod 4$.
   Case 3: If $a \equiv 2 \pmod 4$ then $a = 4k+2$ for $k \in \mathbb{Z}$ and so $a^2 = 16k^2 + 16k + 4 = 4(4k^2+4k+1)$ and so $a^2 \equiv 0 \pmod 4$.
   Case 4: If $a \equiv 3 \pmod 4$ then $a = 4k+3$ for $k \in \mathbb{Z}$ and so $a^2 = 16k^2 + 24k + 9 = 4(4k^2+6k+2)+1$ and so $a^2 \equiv 1 \pmod 4$.
   We see in all cases that $a \not\equiv 2, 3 \pmod 4$.