MATH 310: Groupwork 9 Solutions **Thursday 7/19/2012**

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1. Let $m \in \mathbb{Z}$. Prove if $3 \nmid m$ then $3 \nmid m^2$.
   Proof: We proceed by cases. Since $3 \nmid m$ we know either $m = 3k+1$ or $m = 3k+2$ for some $k \in \mathbb{Z}$.
   Case 1: If $m = 3k+1$ then $m^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ so $3 \nmid m^2$.
   Case 1: If $m = 3k+2$ then $m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ so $3 \nmid m^2$.
   $\ddot\smile$

2. Let $a \in \mathbb{Z}$. Prove that $a^3 \equiv a \pmod 3$.
   Proof: We proceed by cases. Either $a = 3k$, $a = 3k+1$ or $a = 3k+2$ for some $k \in \mathbb{Z}$.
   Case 1: If $m = 3k$ then $a^3 - a = 27k^3 - 3k = 3(9k^3 - k)$ so that $3|(a^3 - a)$ and so $a^2 \equiv a \pmod 3$.
   Case 2: If $m = 3k+1$ then $a^3 - a = (3k+1)^3 - (3k+1) = 27k^3 + 27k^2 + 6k = 3(9k^3 + 9k^2 + 2k)$ so that $3|(a^3 - a)$ and so $a^2 \equiv a \pmod 3$.
   Case 3: Is like case 2, mutatis mutandi.
   $\ddot\smile$

3. Show that if $a \in \mathbb{Z}$ then $a^2 \not\equiv 2 \pmod 4$ and $a^2 \not\equiv 3 \pmod 4$.
   Proof: We proceed by cases. Either $a = 4k$, $a = 4k+1$, $a = 4k+2$ or $a = 4k+3$ for some $k \in \mathbb{Z}$.
   Case 1: If $m = 4k$ then $a^2 = 16k^2 = 4(4k^2)$ so $4|a^2$ so $a^2 \equiv 0 \pmod 4$.
   Case 2: If $m = 4k+1$ then $a^2 = 16k^2 + 8k + 1 = 4(4k^2 + 2k) + 1$ so $a^2 - 1 = 4(4k^2 + 2k)$ so $4|a^2 - 1$ and so $a^2 \equiv 1 \pmod 4$.
   Case 3: If $m = 4k+2$ then $a^2 = 16k^2 + 16k + 4 = 4(4k^2 + 4k + 1)$ so $4|a^2$ and so $a^2 \equiv 0 \pmod 4$.
   Case 4: If $m = 4k+3$ then $a^2 = 16k^2 + 24k + 9 = 4(4k^2 + 6k + 2) + 1$ so $a^2 - 1 = 4(4k^2 + 6k)$ so $4|a^2 - 1$ and so $a^2 \equiv 1 \pmod 4$.
   $\ddot\smile$

4. Prove that If $A \subseteq B$ then $A \cap B = A$.
   Proof: Assume $A \subseteq B$. To show $A \cap B = A$ we need to show $A \cap B \subseteq A$ and $A \subseteq A \cap B$.
   $A \cap B \subseteq A$: Let $x \in A \cap B$ then $x \in A$ by defintion of $\cap$.
   $A \subseteq A \cap B$: Let $x \in A$ then since $A \subseteq B$ we know $x \in B$ and so $x \in A \cap B$.
   $\ddot\smile$

5. Prove that $A \cup B = A \cap B$ iff $A = B$.
   Proof: First we prove that $A \cup B = A \cap B \;\rightarrow\; A = B$ and then we show the reverse.
   $A \cup B = A \cap B \;\rightarrow\; A = B$: Assume $A \cup B = A \cap B$. Let $x \in A$ then $x \in A \cup B$ and so $x \in A \cap B$ and so $x \in B$. Simililary let $x \in B$ then $x \in A \cup B$ and so $x \in A \cap B$ and so $x \in A$. Thus $A = B$.
   $A = B \;\rightarrow A \cup B = A \cap B$: If $A = B$ then of course $A \cup B = A \cup A = A = A \cap A = A \cap B$.
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