1. Prove that $\Big|[0,1]\times[0,1]\Big| = \Big|[0,1]\Big|$ by explicitly finding a bijection between the sets and proving it is a bijection. [10 pts]

   Solution: We'll define a bijection $f : [0,1] \to [0,1]\times[0,1]$. For each $a \in [0,1]$ we can write $a = 0.a_1a_2a_3a_4...$ where the a_i are the digits. Note that $1 = 0.9999...$ and for any other number with multiple representations, like $0.342999... = 0.323000...$ we use the former representation. Then define
   $$f(0.a_1a_2a_3a_4...) = (0.a1a3a5..., 0.a2a4a6...)$$

   Proof: We need to check it's a bijection:

   - Surjectivity: For $(b,c) \in [0,1]\times[0,1]$ we represent these with their decimal expansions: $(b,c) = (0.b_1b_2..., 0.c_1c_2...)$ and then if $a = 0.b_1c_1b_2c_2...$ then $a \in [0,1]$ and $f(a) = (b,c)$ as desired.
   - Injective: Suppose $f(a) = f(b)$ meaning $f(0.a_1a_2...) = f(0.b_1b_2...)$ and so $(0.a_1a_3..., 0.a_2a_4...) = (0.b_1b_3..., 0.b_2b_4...)$ and so all the digits match and $a = b$.

   $\mathcal{QED}$

2. Let A and B be nonempty sets. Prove that $|A| \leq |A\times B|$. [5 pts]

   Proof: Let $b_0 \in B$ be fixed. Observe that $f : A \to A\times B$ given by $f(a) = (a, b_0)$ is an injection.

   $\mathcal{QED}$

3. Find an example of infinite sets A and B with $|A| < |A\times B|$. [5 pts]

   Answer: Let $A = \mathbb{Z}$ and $b = \mathbb{R}$.

4. Find bijections between the following sets. You can use pictures or explicit functions as long as your argument is clear. You do not need to prove bijectivity.

   (a) $\mathbb{Z}$ and $\mathbb{Q}^+$ [10 pts]

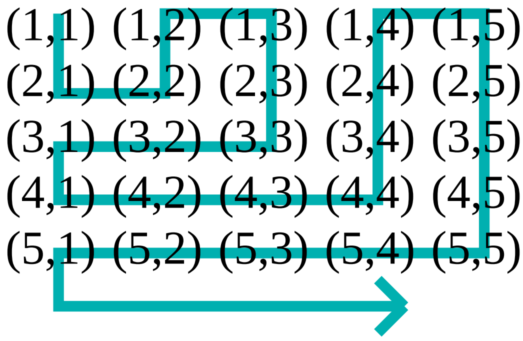
   Solution: We know we can enumerate the integers by $0, 1, -1, 2, -2, 3, -3, ...$ and so we match these value in this order with the snake-diagram from class for enumerating $\mathbb{Q}^+$.

   (b) $\mathbb{Q}^+$ and $\mathbb{Q}$ [10 pts]

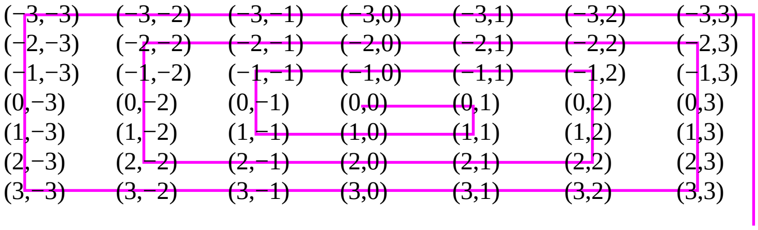
   Solution: We know we can enumerate $\mathbb{Q}^+$ with the snake diagram so we simply start with 0 and then alternate back and forth between positive and negative values from this listing.

(c) $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$ [10 pts]

Solution: We saw how to list $\mathbb{N} \times \mathbb{N}$ on the last homework:



And we can do $\mathbb{Z} \times \mathbb{Z}$ by looping around:



So then what we do is match these together along their orders.

5. Show that $\left\{\frac{3n+1}{9n-1}\right\}$ converges to $\frac{1}{3}$. [10 pts]

Idea: Given ϵ we need to show that there is an N so that for $n > N$ we have

$$\left|\frac{3n+1}{9n-1} - \frac{1}{3}\right| < \epsilon$$
$$\left|\frac{3(3n+1)-(9n-1)}{3(9n-1)}\right| < \epsilon$$
$$\frac{4}{3(9n-1)} < \epsilon$$
$$9n-1 > \frac{4}{3\epsilon}$$
$$n > \frac{4}{27\epsilon} + \frac{1}{9}$$

Proof: For $\epsilon > 0$ we define $N = \lceil \frac{4}{27\epsilon} + \frac{1}{9} \rceil$. Then for $n > N$ we have

$$n > \frac{4}{27\epsilon} + \frac{1}{9}$$
$$\frac{4}{3(9n-1)} < \epsilon$$
$$\left|\frac{3n+1}{9n-1} - \frac{1}{3}\right| < \epsilon$$

QED

6. Show that $\{(-1)^n n^2\}$ does not converge to 3. [10 pts]

Proof: We proceed by contradiction and assume for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ so that $n > N$ implies $|(-1)^n n^2 - 3| < \epsilon$. For n even and greater than 1 this means $n^2 - 3 < \epsilon$ but for $\epsilon = 1$ we get $n^2 < 4$ which contradicts n even and greater than 1.

QED

7. Show that $\left\{\frac{4n^3+n^2+3n+1}{n^3}\right\}$ converges to 4. [10 pts]

Idea: Given ϵ we need to show that there is an N so that for $n > N$ we have

$$\left|4 + \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3} - 4\right| < \epsilon$$
$$\left|\frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3}\right| < \epsilon$$
$$\frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3} < \epsilon$$

For $n \in \mathbb{N}$ we know that

$$\frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3} \leq \frac{1}{n} + \frac{3}{n} + \frac{1}{n} = \frac{5}{n}$$

so so provided we get $\frac{5}{n} < \epsilon$ or $n > \frac{5}{\epsilon}$ we're safe.

Proof: for $\epsilon > 0$ we define $N = \lceil \frac{5}{\epsilon} \rceil$. Then for $n > N$ we have

$$n > \frac{5}{\epsilon}$$
$$\frac{5}{n} < \epsilon$$
$$\frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3} < \frac{5}{n} < \epsilon$$
$$\left|4 + \frac{1}{n} + \frac{3}{n^2} + \frac{1}{n^3} - 4\right| < \epsilon$$

QED

8. Prove that if $\{a_n\}$ is a sequence which converges to a and also to b then $a = b$.
Note: This may seem obvious but the point is to prove it rigorously from the definition. [20 pts]

Idea: If $a \neq b$ we choose an ϵ small enough that a_n cannot be both within ϵ of a and within ϵ of b because a and b are apart from one another.

Proof: We proceed by contradiction and assume that $\{a_n\}$ converges to both a and b with $a \neq b$. Without loss of generality use $b > a$.

By the definition of convergence there is some $N_1 \in \mathbb{N}$ such that for $n > N_1$ we have $|a_n - a| < \frac{b-a}{2}$. This is equivalent to $3a - b < 2a_n < a + b$.
Similarly there is some $N_2 \in \mathbb{N}$ such that for $n > N_2$ we have $|a_n - b| < \frac{b-a}{2}$. This is equivalent to $a + b < 2a_n < 3b - a$.
So then for n greater than both of these we have both being true but then we would have $a + b < 2a_n < a + b$ which is impossible.

QED