MATH 310: Homework 11 Solutions **Due Wednesday 8/15/2012**

1. Suppose $\{a_n\}$ converges to 1. Show that there is some $N \in \mathbb{N}$ such that $(n > N) \to (a_n > 0)$. [5 pts]

   Idea: Eventually the a_n are close to 1 meaning above 0. We just need an appropriate ϵ.

   Proof: Let $\epsilon = 1/2$ then there is some $N \in \mathbb{N}$ such that for $n > N$ we have $|a_n - 1| < 1/2$ which means $-\frac{1}{2} < a_n - 1 < \frac{1}{2}$ or $\frac{1}{2} < a_n < \frac{3}{2}$. The left inequality is the one we desire.

   QED

2. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5 - 3x$ is continuous at $x = 7$. [10 pts]

   Idea: We need to show that if $\{x_n\}$ converges to 7 then $\{f(x_n)\} = \{5 - 3x_n\}$ converges to $f(7) = -16$. This latter convergence would mean that for any $\epsilon > 0$ we can choose an $N \in \mathbb{N}$ so that $n > N$ gives us $|5 - 3x_n - (-16)| < \epsilon$, in other words $3|x_n - 7| < \epsilon$ or $|x_n - 7| < \epsilon/3$. But the convergence of $\{x_n\}$ to 7 means we can choose an N to make this happen.

   Proof: Given ϵ we choose N so that for $n > N$ we have $|x_n - 7| < \epsilon/3$. Then

$$\begin{aligned} |x_n - 7| &< \epsilon/3 \\ 3|x_n - 7| &< \epsilon \\ |5 - 3x_n - (-16)| &< \epsilon \end{aligned}$$

   as desired.

   QED

3. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2 - 2x - 1$ is continuous at $x = 3$. [20 pts]

   Idea: We need to show that if $\{x_n\}$ converges to 3 then $\{f(x_n)\} = \{2x_n^2 - 2x_n - 1\}$ converges to $f(3) = 11$. This latter convergence would mean that for any $\epsilon > 0$ we can choose an $N \in \mathbb{N}$ so that $n > N$ gives us $|2x_n^2 - 2x_n - 1 - 11| < \epsilon$, in other words $|x_n - 3| \cdot |x_n + 2| < \epsilon/2$.

   Now then note that we can make $|x_n - 3|$ as small as we like but we don't immediately have control over $|x_n - 2|$. But really we do, in a way. We can choose $N_1 \in \mathbb{N}$ so that for $n > N_1$ we have $|x_n - 3| < 1$ so that $-1 < x_n - 3 < 1$ which becomes $4 < x_n + 2 < 6$. This will give us $|x_n + 2| < 6$.

   This gives us $|(x_n - 3)(x_n + 2)| < |x_n - 3|(6)$ so what we'll do is make $|x_n - 3|(6) < \epsilon/2$ by making $|x_n - 3| < \epsilon/12$ via a choice of N_2.

   Proof: Given ϵ we choose N_1 so that for $n > N_1$ we have $|x_n - 2| < 1$ and we choose N_2 so that for $n > N_2$ we have $|x_n - 3| < \epsilon/12$. Then let $N = \max\{N_1, N_2\}$.

   The first of these then gives us $|x_n + 2| < 6$ and then together we have

$$|2x_n^2 - 2x_n - 1 - 11| = 2|x_n - 3| \cdot |x_n + 2| < 2|x_n - 3|(6) = 12|x_n - 3| < \epsilon$$

   as desired.

   QED

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by [15 pts]

$$f(x) = \begin{cases} 2 & \text{if } x \leq 3 \\ 1 - x & \text{if } x > 3 \end{cases}$$

Show that f is not continuous at $x = 3$.

Proof by counterexample: Consider that $\{3 + 1/n\}$ converges to 3 but $\{f(3 + 1/n)\} = \{1 - (3 + 1/n)\} = \{-2 - 1/n\}$ converges to -2 which is not $f(3) = 2$.

$\mathcal{QED}$

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 + 3x + 1$. Show that $f'(2) = 7$. [10 pts]

Proof: Suppose $\{x_n\}$ converges to 2. Observe that

$$\left\{ \frac{f(x_n) - f(2)}{x_n - 2} \right\} = \left\{ \frac{(x_n^2 + 3x_n + 1) - 11}{x_n - 2} \right\} = \left\{ \frac{(x_n + 5)(x_n - 2)}{x_n - 2} \right\} = \{x_n + 5\}$$

which converges to 7.

$\mathcal{QED}$

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by [10 pts]

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \leq 2 \\ x^2 - 3 & \text{if } x > 2 \end{cases}$$

Show that $f'(2)$ is undefined.

Proof: We proceed by contradiction and assume that $f'(2) = L$, meaning whenever $\{x_n\}$ converges to 2 we have $\left\{ \frac{f(x_n) - f(2)}{x_n - 2} \right\}$ converging to L.

Now then, consider the two sequences $\{2 - 1/n\}$ and $\{2 + 1/n\}$. Both converge to 2 but:
For $x_n = 2 + 1/n$:

$$\left\{ \frac{f(x_n) - f(2)}{x_n - 2} \right\} = \left\{ \frac{(2 + 1/n)^2 - 3 - 1}{2 + 1/n - 2} \right\} = \left\{ \frac{1}{n} + 4 \right\}$$

which converges to 1, implying $L = 1$.
For $x_n = 2 - 1/n$:

$$\left\{ \frac{f(x_n) - f(2)}{x_n - 2} \right\} = \left\{ \frac{\frac{1}{2}(2 - 1/n) - 1}{2 - 1/n - 2} \right\} = \left\{ \frac{1}{2} \right\}$$

which converges to $\frac{1}{2}$, implying $L = \frac{1}{2}$.
Since $1 \neq \frac{1}{2}$ we have a contradiction.

$\mathcal{QED}$