MATH 310: Homework 3 Solutions

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1. Suppose $P(A) : A \cap \{1,3\} \neq \emptyset$ and $Q(A) : |A - \{1\}| = 2$. For which $A \in \mathcal{P}(\{1,2,3,4\})$ is the biconditional $P(A) \leftrightarrow Q(A)$ a true statement? Justify your steps, don't just give the answer.
   Solution: We have:
   $P(A)$ true for $A \in \{\{1\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{1,2,4\},$
   $\{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$
   $\sim P(A)$ true for $A \in \{\emptyset, \{2\}, \{4\}, \{2,4\}\}$.
   We have:
   $Q(A)$ true for $A \in \{\{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$
   $\sim Q(A)$ false for $A \in \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3,4\}, \{1,2,3,4\}\}$

   The biconditional is true when either $P(A) \wedge Q(A)$ or $\sim P(a) \wedge \sim Q(A)$. The first is true when $A \in \{\{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3\}, \{3,4\}\}$ and the second is true when $A \in \{\emptyset, \{2\}, \{4\}\}$.

   Therefore the biconditional is true when $A \in \{\emptyset, \{2\}, \{4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3\}, \{3,4\}\}$

2. For statements P and Q show that $((P \to Q) \wedge (Q \to R)) \to (P \to R)$ is a tautology.
   Solution: The truth table is as follows:

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$((P \to Q) \wedge (Q \to R)) \to (P \to R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

3. Determine with justification if the following are true or false.

   (a) $\forall n \in \mathbb{Z}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.
   Result: False. For example if $n = 0$ then $\frac{1}{3}(1-2) \notin \mathbb{Z}$.

   (b) $\exists n \in \mathbb{Z}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.
   Result: True. For example if $n = 2$ then $\frac{1}{3}(2-2) \in \mathbb{Z}$.

   (c) $\exists! n \in \mathbb{Z}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.
   Result: False. For example both $n = 2$ and $n = 5$ yield integers.

   (d) $\exists! n \in \{0,1,2,3,4\}$, $\frac{1}{3}(n-2) \in \mathbb{Z}$.
   Result: True. Only $n = 2$ yields an integer.

   (e) $\forall x \in \mathbb{R}$, $x^2 + 3 \geq 0$.
   Result: True. Since $x^2 \geq 0$ we know $x^2 + 3 \geq 3 > 0$.

   (f) $\exists x \in \mathbb{R}$, $x^2 + 3 \geq 0$.
   Result: True. Since every x will work, any x will.

   (g) $\forall x \in \{1,2,3\}$, $3x + 1$ is prime.
   Result: False. For example if $x = 1$ then $3(1) + 1 = 4$ is not prime.

   (h) $\exists x \in \{1,2,3\}$, $3x + 1$ is prime. True. For example if $x = 2$ then $3(2) + 1 = 7$ is prime.
   Result:

   (i) $\exists! x \in \{1,2,3\}$, $3x + 1$ is prime. True. Only $x = 1$ gives a prime number.

4. Fill in the following truth table for all possible values of P, Q and R.

P	Q	R	$P \wedge R$	$Q \to (P \wedge R)$	$(Q \to P) \wedge R$	$R \vee (P \to Q)$
T	T	T	T	T	T	T
T	T	F	F	F	F	T
T	F	T	T	T	T	T
F	T	T	F	F	F	T
T	F	F	F	T	F	F
F	T	F	F	F	F	T
F	F	T	F	T	T	T
F	F	F	F	T	F	T

5. Distribute the negation signs for each of the following, adjusting other symbols accordingly.

   (a) $\sim (\forall x, P(x) \wedge P(x+1))$
   Solution: $\sim (\forall x, P(x) \wedge P(x+1)) = \exists x, \sim P(x) \vee \sim P(x+1)$

   (b) $\sim (\exists x, Q(x) \to Q(x+1))$
   Solution: $\sim (\exists x, Q(x) \to Q(x+1)) = \forall x, Q(x) \wedge \sim Q(x+1)$

   (c) $\sim (\exists x, \forall y P(x,y) \vee Q(x,y))$
   Solution: $\sim (\exists x, \forall y P(x,y) \vee Q(x,y)) = \forall x, \exists y, \sim P(x,y) \wedge \sim Q(x,y)$

   (d) $\sim (\forall x, \exists y P(x,y) \wedge Q(x,y))$
   Solution: $\sim (\forall x, \exists y P(x,y) \wedge Q(x,y)) = \exists x, \forall y, \sim P(x,y) \vee \sim Q(x,y)$

   (e) $\sim (\forall x, \exists y P(x,y) \leftrightarrow Q(x,y))$
   Solution: $\sim (\forall x, \exists y P(x,y) \leftrightarrow Q(x,y)) = \exists x, \forall y, (P(x,y) \wedge \sim Q(x,y)) \vee (\sim P(x,y) \wedge Q(x,y))$

6. Assume a_n is a sequence of real numbers. The formal definition that a_n converges to $a_0 \in \mathbb{R}$ as $n \to \infty$ is:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, (n \geq N \to |a_n - a_0| < \epsilon)$$

   Negate this statement.
   Solution: $\sim [\forall \epsilon > 0, \exists N \in \mathbb{N}, (n \geq N \to |a_n - a_0| < \epsilon)] = \exists \epsilon > 0, \forall N \in \mathbb{N}, (n \geq N \wedge |a_n - a_0| \geq \epsilon)$

7. Negate the following, writing your results in english:

   (a) There was once a year in which every day was rainy or snowy.
   Solution: Every year has one day which is neither rainy nor snowy.

   (b) For every week there is at least one day where if it's cloudy then it snows.
   Solution: There was once a week in which all days were cloudy and did not snow.

8. Provide proofs with justification of each of the following. Some statistics to help:

- One is trivially true.
- One is vacuously true.
- Two should have direct proofs.
- Two should have proofs of the contrapositive.
- One requires an intermediate step by the contrapositive with a link to a direct proof.
- One requires cases.

(a) If $n, m \in \mathbb{Z}$ are both odd then $3n - m + 1$ is odd.
Proof: Suppose both are odd, then $n = 2k + 1$ and $m = 2l + 1$ with $k, l \in \mathbb{Z}$. Then $3n - m + 1 = 3(2k + 1) - (2l + 1) + 1 = 6k - 2l + 1 = 2(3k - l) + 1 = 2b + 1$ with $b = 3k - l \in \mathbb{Z}$ so $3n - m + 1$ is odd. $\ddot\smile$

(b) If $n \in \mathbb{Z}$ and $3n - 7$ is odd then $\frac{n}{2} + 1 \in \mathbb{Z}$.
Lemma: If $3n - 7$ is odd then n is even.
Proof of Lemma: Done in class.
Proof of Problem: If $3n - 7$ is odd then by the lemma n is even and so $n = 2b$ for $b \in \mathbb{Z}$. Then $\frac{n}{2} + 1 = b + 1 \in \mathbb{Z}$ as desired. $\ddot\smile$

(c) If $x \in \mathbb{R}$ and $x^2 + 2x \leq 3$ then $-3 \leq x \leq 1$.
Proof: Suppose $x^2 + 2x \leq 3$. Then $(x + 3)(x - 1) \leq 0$ so either we have $x + 3 \geq 0$ and $x - 1 \leq 0$ or we have $x + 3 \leq 0$ and $x - 1 \geq 0$. In the former case $x \geq -3$ and $x \leq 1$ yielding $-3 \leq x \leq 1$. In the latter case $x \leq -3$ and $x \geq 1$ which is impossible. Thus together we have $-3 \leq x \leq 1$. $\ddot\smile$

(d) If $x \in \mathbb{R}$ and $|x + 1| + 1 = 0$ then $x^2 = 4$.
Proof: Vacuously true since $|x + 1| + 1 = 0$ is never true. $\ddot\smile$

(e) If $n \in \mathbb{Z}$ and $3n + 1$ is odd then n is even.
Proof: We prove the contrapositive, that if n is odd then $3n + 1$ is even. If n is odd then $n = 2k + 1$ for $k \in \mathbb{Z}$. Then $3n + 1 = 3(2k + 1) + 1 = 6k + 4 = 2(3k + 2) = 2b$ for $b = 3k + 2 \in \mathbb{Z}$ and so $3n + 1$ is even. $\ddot\smile$

(f) If $n \in \mathbb{Z}$ and $n^2 + n < 0$ then $|n + 1| + 1 > 0$.
Proof: Trivially true since $|n + 1| + 1 > 0$ for all $n \in \mathbb{Z}$. $\ddot\smile$

(g) If $n \in \mathbb{Z}$ then $n^2 + n + 1$ is odd.
Proof: We examine the cases where n is even and odd.
Case 1: If n is even then $n = 2k$ for $k \in \mathbb{Z}$ so then $n^2+n+1 = (2k)^2+2k+1 = 2(2k^2+k)+1$ so $n^2 + n + 1$ is odd.
Case 2: If n is odd then $n = 2k+1$ for $k \in \mathbb{Z}$ so then $n^2+n+1 = (2k+1)^2+2k+1+1 = 2(2k^2 + 3k + 1) + 1$ so $n^2 + n + 1$ is odd. $\ddot\smile$

(h) If $f(x)$ is a function and $f'(x) - 2f(x) = 0$ then $f(x) \neq \sin(2x)$.
Proof: We prove the contrapositive, tht if $f(x) = \sin(2x)$ then $f'(x) - 2f(x) \neq 0$. If $f(x) = \sin(2x)$ then $f'(x) - 2f(x) = 2\cos(2x) - \sin(2x)$ and this does not equal 0, for example, when $x = 0$. $\ddot\smile$

9. Explain why the following proofs fail. Explanations should be in full sentences with minimal notation.

   (a) Claim: If $x^2 - 4 = 0$ then $x = 2$.
   "Proof": Suppose $x = 2$. Then $x^2 = 4$ and so $x^2 - 4 = 0$.
   Problem: The "proof" is actually of the converse.

   (b) Claim: $3 = -3$.
   "Proof": Let $x = 3$. Then $x^2 = (-x)^2$ so $\sqrt{x^2} = \sqrt{(-x)^2}$ and so canceling the square root and the square yields $x = -x$ and so $3 = -3$.
   Problem: The equation $\sqrt{x^2} = x$ is only valid for positive x and so $\sqrt{(-3)^2} \neq -3$.

   (c) Claim: $1 = -1$.
   "Proof": $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$.
   Problem: The equation $\sqrt{xy} = \sqrt{x}\sqrt{y}$ is only valid for positive x, y and so $\sqrt{(-1)(-1)} \neq \sqrt{-1}\sqrt{-1}$.