MATH 310: Homework 4 Solutions **Due Monday 7/23/2012**

1. Let $a, b \in \mathbb{Z}$. Prove that if $a|b$ and $b|a$ then $a = \pm b$.

   Proof: First we prove $\rightarrow$ and then $\leftarrow$.

   Part 1: If $a|b$ and $b|a$ then $a = \pm b$: Since $a|b$ and $b|a$ we know $a = bm$ and $b = an$ for $m, n \in \mathbb{Z}$. Then $a = bm = anm$ and so $nm = 1$ and either $n = m = 1$ or $n = m = -1$. In the former case $a = b$ and in the latter case $a = -b$.

   Part 2: If $a = \pm b$ then $a|b$ and $b|a$: If $a = \pm b$ then $a = \pm 1 \cdot b$ and $b = \pm 1 \cdot a$ so $a|b$ and $b|a$. $\ddot\smile$

2. Let $x, y \in \mathbb{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$ then $3|(x^2 - y^2)$.

   Proof: There are four cases to deal with because we could have $x = 3k + 1$ or $x = 3k + 2$ for $k \in \mathbb{Z}$ and we could have $y = 3n + 1$ or $y = 3n + 2$ for $n \in \mathbb{Z}$.

   Case 1: If $x = 3k + 1$ and $y = 3n + 1$ then $x^2 - y^2 = (3k + 1)^2 - (3n + 1)^2 = 9kn + 6k - 6n = 3(3kn + 2k - 2n)$ and so $3|(x^2 - y^2)$.

   Case 2: If $x = 3k + 1$ and $y = 3n + 2$ then $x^2 - y^2 = (3k + 1)^2 - (3n + 2)^2 = 9kn + 6k - 12n - 3 = 3(3kn + 2k - 4n - 1)$ and so $3|(x^2 - y^2)$.

   Case 3: If $x = 3k + 2$ and $y = 3n + 1$ then $x^2 - y^2 = (3k + 2)^2 - (3n + 1)^2 = 9kn + 12k - 6n + 3 = 3(3kn + 4k - 2n + 1)$ and so $3|(x^2 - y^2)$.

   Case 4: If $x = 3k + 2$ and $y = 3n + 2$ then $x^2 - y^2 = (3k + 2)^2 - (3n + 2)^2 = 9kn + 12k - 12n = 3(3kn + 4k - 4n)$ and so $3|(x^2 - y^2)$. $\ddot\smile$

3. Show that if a is an odd integer then $a^2 \equiv 1 \pmod 8$.

   Pre-Proof Note: If $a = 2k + 1$ for $k \in \mathbb{Z}$ then $a^2 - 1 = 4k^2 + 4k$ which gives us $a^2 \equiv 1 \pmod 4$, not good enough. Instead we could divide a by 8 instead of 2 but it suffices to divide by 4.

   Proof: Since a is odd we know either $a = 4k + 1$ or $a = 4k + 3$ with $k \in \mathbb{Z}$.

   Case 1: If $a = 4k + 1$ then $a^2 - 1 = 16k^2 + 8k = 8(2k^2 + 1)$ so $8|(a^2 - 1)$ and $a^2 \equiv 1 \pmod 8$.

   Case 1: If $a = 4k + 3$ then $a^2 - 1 = 16k^2 + 24k = 8(2k^2 + 3)$ so $8|(a^2 - 1)$ and $a^2 \equiv 1 \pmod 8$. $\ddot\smile$

4. Let $m, n \in \mathbb{Z}$. Prove that if $n \equiv 1 \pmod 2$ and $m \equiv 3 \pmod 4$ then $n^2 + m \equiv 0 \pmod 4$.

   Proof: Since $n \equiv 1 \pmod 2$ we know $2|(n - 1)$ so $n = 2k + 1$ for $k \in \mathbb{Z}$. Likewise since $m \equiv 3 \pmod 4$ then $4|(m - 3)$ so $m = 4n + 3$ for $n \in \mathbb{Z}$. Then $n^2 + m = 4k^2 + 4k + 1 + 4n + 3 = 4(k^2 + k + n + 1)$ and so $n^2 + m \equiv 0 \pmod 4$. $\ddot\smile$

5. Let $x, y \in \mathbb{R}$. Prove thave if $x^2 - 4x = y^2 - 4y$ and $x \neq y$ then $x + y = 4$.

   Proof: Given that $x^2 - 4x = y^2 - 4y$ we add 4 to both sides and factor, yielding $(x-2)^2 = (y-2)^2$. From here we know either $x - 2 = +(y-2)$ or $x - 2 = -(y-2)$. The former case gives us $x = y$ which we know is not true so we have the latter case which gives us $x - 2 = 2 - y$ so $x + y = 4$. $\ddot\smile$

6. Let A and B be sets. Prove that $A \cap B = A$ iff $A \subseteq B$.

   Proof: First we show that $A \cap B = A \ \rightarrow \ A \subseteq B$ and then we show the reverse.

   Part 1: $A \cap B = A \ \rightarrow \ A \subseteq B$: Let $x \in A$. Since $a = A \cap B$ we then have $x \in A \cap B$ so $x \in B$ as desired.

   Part 2:
   $A \subseteq B \ \rightarrow \ A \cap B = A$: We need to show $A \cap B \subseteq A$ and $A \subseteq A \cap B$. First let $x \in A \cap B$. Since $A \cap B \subseteq A$ we have $x \in A$ as desired. Second let $x \in A$. Since $A \subseteq B$ we know $x \in B$ and so $x \in A \cap B$ as desired. $\ddot\smile$

7. Let A and B be sets. Prove that $A \cup B = A \cap B$ iff $A = B$.

   Proof: First we prove that $A \cup B = A \cap B \ \rightarrow \ A = B$ and then we show the reverse.

   Part 1: $A \cup B = A \cap B \ \rightarrow \ A = B$: Assume $A \cup B = A \cap B$. Let $x \in A$ then $x \in A \cup B$ and so $x \in A \cap B$ and so $x \in B$. Simililary let $x \in B$ then $x \in A \cup B$ and so $x \in A \cap B$ and so $x \in A$. Thus $A = B$.

   Part 2: $A = B \ \rightarrow A \cup B = A \cap B$: If $A = B$ then of course $A \cup B = A \cup A = A = A \cap A = A \cap B$. $\ddot\smile$