1. For each of the following indicate symoblically what you would assume for each proof method: Direct, by contrapositive and by contradiction.

   (a) $\forall x, P(x) \to Q(x)$

   Solution:
   Direct: $P(x)$
   Contrapositive: $\sim Q(x)$
   Contradiction: $\exists x, P(x) \wedge (\sim Q(x))$

   (b) $P \to (Q \vee R)$

   Solution:
   Direct: P
   Contrapositive: $(\sim Q) \wedge (\sim R)$.
   Contradiction: $P \wedge ((\sim Q) \wedge (\sim R))$

   (c) $\forall x, \exists y, P(x,y) \to (Q(x,y) \wedge R(x,y))$

   Solution:
   Direct: $P(x,y)$
   Contrapositive: $(\sim Q(x,y)) \vee (\sim R(x,y))$
   Contradiction: $\exists x, \forall y, P(x,y) \wedge ((\sim Q(x,y)) \vee (\sim R(x,y)))$

   (d) $\forall x, (P(x) \vee Q(x)) \to R(x)$

   Solution:
   Direct: $P(x) \vee Q(x)$
   Contrapositive: $\sim R(x)$
   Contradiction: $\exists x, (P(x) \vee Q(x)) \wedge (\sim R(x))$

   (e) $\forall x, P(x) \vee (Q(x) \to R(x))$

   Solution: Note this was trickier than intended since it's not an implication as written and we need to rewrite it. Without the x for clarity:

   $$P \vee (Q \to R) = P \vee ((\sim Q) \vee R) = (P \vee (\sim Q)) \vee R = \sim (P \vee (\sim Q)) \to R$$

   Direct: $\sim (P(x) \vee (\sim Q(x)))$
   Contrapositive: $\sim R(x)$
   Contradiction: $\exists x, (\sim (P(x) \vee (\sim Q(x)))) \wedge (\sim R(x))$

2. Prove that for $a, b, c \in \mathbb{Z}$ that if $a|(b+c)$ and $a \nmid b$ then $a \nmid c$.

   Proof: We proceed by contradiction assuming that there are $a, b, c \in \mathbb{Z}$ with $a|(b+c)$, $a \nmid b$ and $a|c$. Then $ak = b + c$ for $k \in \mathbb{Z}$ and $aj = c$ for $j \in \mathbb{Z}$. Then $ak = b + c = b + aj$ so $ak - aj = b$ so $a(k-j) = b$ and so $a|b$, a contradiction. $\mathcal{QED}$

3. Prove that if a and b are odd integers then $4 \nmid (a^2 + b^2)$.

   Proof: We proceed by contradiction assuming that a and b are odd and $4 \mid (a^2 + b^2)$. We have $a = 2k + 1$ and $b = 2j + 1$ for $j, k \in \mathbb{Z}$. Then

   $$a^2 + b^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 = 4(k^2 + k + j^2 + j) + 2$$

   so $4 \nmid (a^2 + b^2)$, a contradiction. $\mathcal{QED}$
   Note: This could be done directly and would be essentially the same.

4. Prove that the sum of the two legs of a right triangle must be greater than the hypotenuse.

   Proof: We proceed by contradiction assuming we have a right triangle with legs a and b and hypotenuse c such that $c \geq a + b$. Squaring both sides and applying the Pythagorean Theorem yields

   $$c^2 \geq (a + b)^2 = a^2 + 2ab + b^2 = c^2 + 2ab$$

   from whence it follows that $2ab \leq 0$ which is impossible since $a, b > 0$. $\mathcal{QED}$

5. Prove that $\sqrt{3}$ is irrational. Just as with our proof with $\sqrt{2}$ you will need a lemma. State and prove this lemma as part of your solution.

   Lemma: For $a \in \mathbb{Z}$ we have $3|a^2$ iff $3|a$.

   Proof: First we prove if $3|a$ then $3|a^2$: If $3|a$ then $a = 3k$ for $k \in \mathbb{Z}$ so then $a^2 = 9k^2 = 3(3k^2)$ so $3|a^2$.
   Next we prove if $3|a^2$ then $3|a$ by proving the contrapositive, if $3 \nmid a$ then $3 \nmid a^2$. There are two cases if $3 \nmid a$. First, if $a = 3k + 1$ for $k \in \mathbb{Z}$ then $a^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ so $3 \nmid a^2$. Second, if $a = 3k + 2$ for $k \in \mathbb{Z}$ then $a^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ so $3 \nmid a^2$.

   Proof of Problem: We proceed by contradiction, assuming that $\sqrt{3}$ is rational. If so, then $\sqrt{3} = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ in lowest terms. Then we have $3 = \frac{a^2}{b^2}$ and so $3b^2 = a^2$ which tells us $3|a^2$ so $3|a$ by the lemma. This then gives us $a = 3k$ for $k \in \mathbb{Z}$ and so $3b^2 = (3k)^2 = 9k^2$ so $b^2 = 3k^2$ so $3|b^2$ and so $3|b$. But we cannot have $3|a$ and $3|b$ since $\frac{a}{b}$ is in lowest terms. $\mathcal{QED}$

6. Prove there does not exist a real number x such that $x^6 + x^4 + 1 = 2x^2$.

   Proof: Assume by way of contradiction that we do. Then

   $$\begin{aligned} x^6 + x^4 + 1 &= 2x^2 \\ x^6 + x^4 - 2x^2 + 1 &= 0 \\ x^6 + (x^2 - 1)^2 &= 0 \end{aligned}$$

   Since both summands have even powers both are nonnegative. But since the sum is zero they both must equal zero. The first tells us $x = 0$ while the second tells us $x = \pm 1$. This is a contradiction. $\mathcal{QED}$

7. Prove that the equation $x^3 + x + 1 = 0$ has a real solution but no rational solution.
Hint: For the second part, if $\frac{p}{q}$ is such a root in lowest terms examine the parities of p and q.

Proof: First observe that when $x = 0$ we have $x^3 + x + 1 = 1 > 0$ and when $x = -1$ we have $x^3 + x + 1 = -1$ so, since $x^3 + x + 1$ is continuous, by the Intermediate Value Theorem we know there is an $x \in (-1, 0)$ with $x^3 + x + 1 = 0$.
For the second part we proceed by contradiction. Assume that $x = \frac{a}{b}$ is a rational solution with $a, b \in \mathbb{Z}$ in lowest terms. So then we have

$$x^3 + x + 1 = 0$$
$$\left(\frac{a}{b}\right)^3 + \left(\frac{a}{b}\right) + 1 = 0$$
$$a^3 + ab^2 + b^3 = 0$$

Now consider the parities of a and b. Since $\frac{a}{b}$ is in lowest terms they're not both even and so we have three cases with $j, k \in \mathbb{Z}$:
Case 1: $a = 2j + 1$ and $b = 2j + 1$: Then $a^3 + ab^2 + b^3 = 2(4j^3 + 6j^2 + 4j + 4jk^2 + 4jk + 8k^2 + 5k + 4k^3 + 1) + 1$ which is odd and hence not zero.
Case 2: $a = 2j + 1$ and $b = 2j$: Then $a^3 + ab^2 + b^3 = 2(4j^3 + 6j^2 + 3j + 4jk^2 + 2k^2 + 4k^3) + 1$ which is odd and hence not zero.
Case 3: $a = 2j$ and $b = 2j + 1$: Then $a^3 + ab^2 + b^3 = 2(4j^3 + 4jk^2 + 4jk + j + 4k^3 + 6k^2 + k) + 1$ which is odd and hence not zero. $\mathcal{QED}$

8. Suppose I have a list of real numbers, all between 0 and 1, listed with decimal expansion as follows, with each variable representing a digit:

$$0.a_{11}a_{12}a_{13}...$$
$$0.a_{21}a_{22}a_{23}...$$
$$0.a_{31}a_{32}a_{33}...$$
$$...$$

Prove there exists a real number not in the list.

Proof: Construct a new number $0.a_1a_2...$ as follow:
Let a_1 be a digit which is not a_{11}.
Let a_2 be a digit which is not a_{22}.
Let a_3 be a digit which is not a_{33}.
And so on. Then clearly this new number is not in the list since it differs from the n^{th} number at the n^{th} digit. $\mathcal{QED}$

9. Prove that there are infinitely many $x, y \in \mathbb{Z}$ with $4x - 6y = 14$.

Proof: We see a pattern in the solutions:
$x = 5, y = 1$
$x = 8, y = 3$
$x = 11, y = 5$ And so it looks like $x = 5 + 3t$, $y = 1 + 2t$ for $t \in \mathbb{Z}$ works. We check:

$$4(5 + 3t) - 6(1 + 2t) = 20 + 12t - 6 - 12t = 14$$

and it works. $\mathcal{QED}$