MATH 310: Homework 6 Solutions **Due Monday 7/30/2012**

1. Suppose that n is a positive even integer with $\frac{n}{2}$ odd. Prove that there do not exist positive integers x and y with $x^2 - y^2 = n$. [10 pts]

   Proof: Assume by way of contradiction that n is a positive even integer with $\frac{n}{2}$ odd and that x and y are positive integer solutions to $x^2 - y^2 = n$. Since $\frac{n}{2}$ is odd we have $\frac{n}{2} = 2x + 1$ for $x \in \mathbb{Z}$ and so $n = 4x + 2$.
   Consider now the parities of x and y:
   Case 1: If $x = 2j$ and $y = 2k$ for $j, k \in \mathbb{Z}$ then $x^2 - y^2 = 4j^2 - 4k^2 = 4(j^2 - k^2)$.
   Case 2: If $x = 2j+1$ and $y = 2k$ for $j, k \in \mathbb{Z}$ then $x^2 - y^2 = 4j^2 + 4j + 1 - 4k^2 = 4(j^2 + j - k^2) + 1$
   Case 3: If $x = 2j$ and $y = 2k + 1$ for $j, k \in \mathbb{Z}$ then $x^2 - y^2 = 4j^2 - 4k^2 - 4k - 1 = 4(j^2 - k^2 - k - 1) + 3$.
   Case 4: If $x = 2j + 1$ and $y = 2k + 1$ for $j, k \in \mathbb{Z}$ then $x^2 - y^2 = 4j^2 + 4j + 4k^2 + 4k = 4(j^2 + j + k^2 + k)$ In each case we do not get $4x + 2$, a contradiction. $\mathcal{QED}$

2. Let $m \in \mathbb{Z}$. Prove that if $3 \nmid (m^2 - 1)$ then $3|m$ by contradiction. [10 pts]

   Proof: Assume by way of contradiction that $3 \nmid (m^2 - 1)$ but $3 \nmid m$. Then we have two cases:
   Case 1: If $m = 3k + 1$ for $k \in \mathbb{Z}$ then $m^2 - 1 = 9k^2 + 6k = 3(3k^2 + 2k)$, contradicting $3 \nmid (m^2 - 1)$.
   Case 2: If $m = 3k = 2$ for $k \in \mathbb{Z}$ then $m^2 - 1 = 9k^2 + 12k + 3 = 3(3k^2 + 6k + 1)$, contradicting $3 \nmid (m^2 - 1)$. $\mathcal{QED}$

3. Let x be a positive real number. Prove that if $x - \frac{2}{x} > 1$ then $x > 2$: [30 pts]

   (a) By a direct proof.

   Proof: Assume $x - \frac{2}{x} > 1$ then $x^2 - 2 > x$ and so $x^2 - x - 2 > 0$ which factors to give $(x - 2)(x + 1) > 0$. We then have two cases:
   Case 1: $x - 2 > 0$ and $x + 1 > 0$: These combine to yield $x > 2$.
   Case 2: $x - 2 < 0$ and $x + 1 < 0$: These combine to yield $x < -1$ which is impossible since x is a positive real number. $\mathcal{QED}$

   (b) By proving the contrapositive.

   Proof: Assume $x \leq 2$ and then observe that $\frac{2}{x} \geq 1$ and so $x - \frac{2}{x} \leq 2 - 1 = 1$. $\mathcal{QED}$

   (c) By contradiction.

   Proof: Assume $x - \frac{2}{x} > 1$ and $x \leq 2$. Then $x - \frac{2}{x} \leq 2 - 1 = 1$ and so $1 \geq x - \frac{2}{x} > 1$, a contradiction. $\mathcal{QED}$

4. Prove that for all $n \in \mathbb{N}$ that $3|(n^3 - n)$: [20 pts]

   (a) Directly with cases.

   Proof: There are three cases we must examine:
   Case 1: If $n = 3k$ for $k \in \mathbb{Z}$ then $n^3 - n = 27k^3 - 3k = 3(9k^3 - k)$.
   Case 2: If $n = 3k + 1$ for $k \in \mathbb{Z}$ then $n^3 - n = 27k^3 + 27k^2 + 6k = 3(9k^3 + 9k^2 + 2k)$
   Case 3: If $n = 3k+2$ for $k \in \mathbb{Z}$ then $n^3 - n = 27k^3 + 54k^2 + 33k + 6 = 3(9k^3 + 18k^2 + 11k + 2)$
   In all three cases we see that $3|(n^3 - n)$. $\mathcal{QED}$

   (b) Using induction.

   Proof: We have:
   Base Case: If $n = 1$ then $n^3 - n = 1^3 - 1 = 0$ and $3|0$.
   Inductive Step: We assume $3|(n^3 - n)$ and prove $3|((n+1)^3 - (n+1))$. The assumption may be rewritten as $n^3 - n = 3k$ for $k \in \mathbb{Z}$ and then observe that

   $$((n+1)^3 - (n+1) = n^3 + 3n^2 + 2n = 3k + n + 3n^2 + 2n = 3(k + n^2 + n)$$

   and we have our conclusion. $\mathcal{QED}$

5. Show that $\exists! x \in \mathbb{R},\ x^5 + 2x^3 + x - 5 = 0$. [10 pts]

   Proof: First observe that when $x = 0$ the expression yields $0^5 + 2(0)^3 + 0 - 5 = -5$ and when $x = 2$ the expression yields $2^5 + 2(5)^3 + 2 - 5 > 0$ so by the Intermediate Value Theorem we are guaranteed an $x \in (0, 2)$ with $x^5 + 2x^3 + x - 5 = 0$.
   Next we must prove uniqueness. Assume by way of contradiction that there are more than one such x. Call two of them x_1 and x_2. Assume without loss of generality that $x_1 < x_2$. By the Mean Value Theorem there exists some $x \in (x_1, x_2)$ with $\frac{d}{dx}\left(x^5 + 2x^3 + x - 5\right) = 0$. But this derivative is $5x^4 + 6x^2 + 1$ which is always positive, a contradiction. $\mathcal{QED}$

6. Prove by induction that $n^3 \leq 3^n$ for $n \geq 4$. [10 pts]

   Proof: We have
   Base Case: When $n = 4$ we check if $4^3 \leq 3^4$ which is true.
   Inductive Step: We assume that $n^3 \leq 3^n$ and we wish to prove $(n+1)^3 \leq 3^{n+1}$. We'll show instead that $3^{n+1} - (n+1)^3 \geq 0$. To see this observe:

   $$3^{n+1} - (n+1)^3 = 3 \cdot 3^n - (n+1)^3 \geq 3n^3 - (n+1)^3 = 2n^3 - 3n^2 - 3n - 1 = n(n(2n-3) - 3)$$

   Now then since $n \geq 4$ we have $2n - 3 \geq 5$ and so $n(n(2n-3) - 3) \geq 4(4(5) - 3) = 68 > 0$ and so we have our claim. $\mathcal{QED}$

7. Prove by induction that $7|(3^{2n} - 2^n)$ for every nonnegative integer n. [10 pts]

   Proof: We have
   Base Case: When $n = 0$ we check if $7|(3^{2(0)} - 2^0)$ which is true.
   Inductive Step: We assume $7|(3^{2n} - 2^n)$ and we prove $7|(3^{2(n+1)} - 2^{n+1})$. To see this first rewrite the assumption as $3^{2n} - 2^n = 7k$ for some $k \in \mathbb{Z}$ and then observe that

   $$\begin{aligned} 3^{2(n+1) - 2^{n+1}} &= 9 \cdot 3^{2n} - 2 \cdot 2^n \\ &= 7^{2n} + 2 \cdot 3^{2n} - 2 \cdot 2^n \\ &= 7^{2n} + 2(7k) \quad \text{By the IH} \\ &= 7\left[6^{2n-1} + 2k\right] \end{aligned}$$

   as desired. $\mathcal{QED}$

8. Define $a_1 = 1$, $a_2 = 4$ and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 3$. Show that $a_n = 3n - 2$ for all $n \in \mathbb{N}$ by strong induction.

   Proof: We have:
   Base Cases: We check if $a_1 = 1 = 3(1) - 2$ which is true and if $a_2 = 4 = 3(2) - 2$ which is true.
   Inductive Step: We assume that $a_i = 3i - 2$ for $1 \leq i \leq n$ and we prove $a_{n+1} = 3(n+1) - 2$. To do this observe that

   $$a_{n+1} = 2a_n - a_{n-1} = 2(3n-2) - (3(n-1) - 2) = 3(n+1) - 2$$

   as desired. $\mathcal{QED}$

9. Prove or disprove each of the following: [20 pts]

   (a) There exists a real number x with $x^2 < x < x^3$.

   Proof: Assume by way of contradiction that there is such an x. The inequality then splits to give us $x^2 < x$ and $x < x^3$. These factor to give:

   $$x(x-1) < 0 \text{ and } x(1-x)(1+x) < 0$$

   Now then there are several cases:
   Case 1: If $x \geq 1$ then $x - 1 \geq 0$ and so $x(x-1) \geq 0$ which contradicts the first.
   Case 2: If $x \leq -1$ then $x - 1 \leq -2$ and so $x(x-1) \geq 2 > 0$ which contradicts the first.
   Case 3: If $0 < x < 1$ then $1 - x > 0$ and $1 + x > 0$ and so $x(1-x)(1+x) > 0$ which contradicts the second.
   Case 4: If $-1 < x < 0$ then $1 - x > 0$ and $1 + x > 0$ and so $x(1-x)(1+x) > 0$ which contradicts the second.
   Either way we have a contradiction and the result is proved. $\mathcal{QED}$

   (b) If A, B, C are sets and $A \cap B = A \cap C$ then $B = C$.

   False, for example if $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 3\}$.

   (c) Every nonzero rational number is the product of two irrational numbers.

   Lemma 1: If $a \in \mathbb{Z}$ then $a\sqrt{2}$ is irrational: Proof: Assume to the contrary that $a\sqrt{2} = \frac{c}{d}$ with $c, d \in \mathbb{Z}$ then $\sqrt{2} = \frac{c}{da}$ which contradicts $\sqrt{2}$ being irrational. $\mathcal{QED}$

   Lemma 2: If $b \in \mathbb{Z}$ then $\frac{1}{b\sqrt{2}}$ is irrational. Proof: Assume to the contrary that $\frac{1}{b\sqrt{2}} = \frac{c}{d}$ with $c, d \in \mathbb{Z}$ then $\sqrt{2} = \frac{d}{bc}$ which contradicts $\sqrt{2}$ being irrational. $\mathcal{QED}$

   Proof of Problem: Let $q \in \mathbb{Q}$ so $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$. Observe that $\left(a\sqrt{2}\right)\left(\frac{1}{b\sqrt{2}}\right) = q$ and by our lemmas these are both irrational. $\mathcal{QED}$

   (d) Every odd integer is the sum of three odd integers.

   Proof: Observe that if n is odd then $n = n + 1 - 1$, all of which are odd. $\mathcal{QED}$