MATH 310: Homework 7 Solutions **Due Thursday 8/2/2012**

1. Define $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$ and define the relation R from A to B by [5 pts]

$$R = \{(a, x), (a, z), (b, y), (b, z), (c, y)\}$$

(a) List the elements in $\{\alpha \in A \mid \alpha R y\}$.

Solution: $\{b, c\}$

(b) Find $|\{(\alpha, \beta) \mid (\alpha = a) \vee (\beta = z)\}|$.

Solution: $|\{(a, x), (a, z), (b, z)\}| = 3$

2. Define a relation R on $\mathbb{Z}$ by $R = \{(a, b) \mid a \leq b + 2\}$. [15 pts]

(a) Prove or disprove: R is reflexive.

Proof: Let $a \in \mathbb{Z}$, than $a \leq a + 2$ and so aRa. $\mathcal{QED}$

(b) Prove or disprove: R is symmetric.

Counterexample: Observe that for $a = 1$ and $b = 4$ we have $1 \leq 4 + 2$ but not $4 \leq 1 + 2$. Hence $1R4$ but $4 \not R 1$.

(c) Prove or disprove: R is transitive.

Counterexample: Observe that for $a = 4$, $b = 2$ and $c = 0$ we have $4 \leq 2 + 2$ and $2 \leq 0 + 2$ but not $4 \leq 0 + 2$. Hence $4R2$ and $2R0$ but $4 \not R 0$.

3. Suppose A is the set of all students in this class and we define a relation R on A by sRt if student s has a birthday before or the same day as t. Let x be you. Find the set $\{y \in A \mid xRy\}$. List by first names only. [10 pts]

Solution: So many people. Full points for everyone!

4. Define a relation R on $\mathbb{Z}$ by $R = \{(a, b) \mid 4|(3a + b)\}$. [20 pts]

(a) Prove that R is an equivalence relation.

Proof: We have three criteria to check:

- Reflexive: For $a \in \mathbb{Z}$ observe that $4|(3a + a)$ and hence aRa.
- Symmetric: For $a, b \in \mathbb{Z}$ suppose aRb which means $4|(3a + b)$ so that $4k = 3a + b$ for some $k \in \mathbb{Z}$. Then observe

$$3b + a = 3(4k - 3a) + a = 12k - 8a = 4(3k - 2a)$$

  so that $4|(3b + a)$ and so bRa.
- Transitive: For $a, b, c \in \mathbb{Z}$ suppose aRb and bRc which means $4k = 3a + b$ and $4n = 3b + c$ for $k, n \in \mathbb{Z}$. Then observe

$$3a + c = 4k - b + (4n - 3b) = 4k - 4b + 4n = 4(k - b + n)$$

  so that $4|(3a + c)$ and so aRc.

$\mathcal{QED}$

(b) List the distict equivalence classes for R.

Solution: The equivalence classes are:
$\{..., -8, -4, 0, 4, 8, ..\}$
$\{..., -7, -3, 1, 5, 9, ...\}$
$\{..., -6, -2, 2, 6, 10, ...\}$
$\{..., -5, -1, 3, 7, 11, ...\}$

Note: If you simply listed these here that's fine but be prepared on the exam to prove this is the case!

5. Define a relation R on $\mathbb{Z}$ by $R = \{(a, b) \mid a^2 + b^2 \text{ is even}\}$. [20 pts]

(a) Prove that R is an equivalence relation.

Proof: We have three criteria to check:

- Reflexive: For $a \in \mathbb{Z}$ observe that $a^2 + a^2 = 2a^2$ so $a^2 + a^2$ is even so aRa.
- Symmetric: For $a, b \in \mathbb{Z}$ suppose aRb which means $a^2 + b^2$ is even but $b^2 + a^2 = a^2 + b^2$ so $b^2 + a^2$ is even so bRa.
- Transitive: For $a, b, c \in \mathbb{Z}$ suppose aRb and bRc which means $a^2 + b^2 = 2k$ and $b^2 + c^2 = 2j$ for $k, j \in \mathbb{Z}$. Then observe $a^2 + c^2 = (2k - b^2) + (2j - b^2) = 2k + 2j - 2b^2 = 2(k + j - b^2)$ so $a^2 + c^2$ is even so aRc.

$\mathcal{QED}$

(b) List the distict equivalence classes for R.

Solution: The equivalence classes are:
$\{..., -3, -1, 1, 3, ...\}$
$\{..., -4, -2, 0, 2, 4, ...\}$

Note: Same comment as previous problem.

6. Let $A = \{1,2,3,4,5,6\}$. Suppose the distict equivalence classes for some relation R are $\{1,4,5\}$, $\{2,6\}$ and $\{3\}$. What is R? [5 pts]

   Solution:

   $$R = \{(1,1),(4,4),(5,5),(1,4),(1,5),(4,5),(4,1),(5,1),(5,4),(2,2),(6,6),(2,6),(6,2),(3,3)\}$$

7. A relation R on a nonempty set A is defined to be *circular* if for all $x, y, z \in A$ we have $(xRy \ \wedge \ yRz) \rightarrow zRx$. Prove that R is an equivalence relation iff R is circular and reflexive. [20 pts]

   Proof: We must show both directions.

   $\rightarrow$ First we show that if R is an equivalence relation then R is circular and reflexive. Assume R is an equivalence relation. We therefore automatically know it's reflexive, we just need to know it's circular. Suppose aRb and bRc. By transitivity we then know aRc but by symmetry we then have cRa. This proves circularity.

   $\leftarrow$ Second we show that if R is circular and reflexive then R is an equivalence relation. Assume R is circular and reflexive. We must show the three requirements for an equivalence relation.

   Reflexive: Automatic since it's given.

   Symmetric: Assume aRb. Since bRb by reflexivity we get bRa by circularity. This establishes symmetry.

   Transitive: Assume aRb and bRc. By circularity we get cRa and by symmetry we get aRc. This establishes transitivity.

   $\mathcal{QED}$