MATH 310: Homework 8 Solutions **Due Monday 8/6/2012**

1. Find counterexamples to the following false claims:
   (a) In $\mathbb{Z}_{10}$ we have $[2][x] = [6] \;\rightarrow\; [x] = [3]$. [5 pts]

   Solution: A counterexample is $[x] = [8]$.

   (b) Delete. [5 pts]

   (c) In $\mathbb{Z}_{15}$ we have $[x][y] = [0] \;\rightarrow\; ([x] = [0] \;\vee\; [y] = [0])$. [5 pts]

   Solution: A counterexample is $[x] = [3]$ and $[y] = [5]$.

2. Write down a multiplication table for the equivalence classes of the integers modulo 5. [10 pts]

   Solution: We have the following:

·	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

3. By passing to equivalence classes modulo a judicious choice of n solve the following problems:
   (a) Show that there are no integer solutions to $21x^3 - 3x^2 + 2x + 8 = 0$. [15 pts]

   Proof: We proceed by contradiction and suppose $x \in \mathbb{Z}$ is a solution. Passing to equivalence classes modulo $n = 5$ we get:

$$\begin{aligned}[21][x]^3 + [-3][x]^2 + [2][x] + [8] &= 0\\ [x]^3 + [2][x]^2 + [2][x] + [3] &= 0\end{aligned}$$

   Since we are in $\mathbb{Z}_5$ there are only five possible choices for $[x]$ so we test them all:

$$\begin{aligned}&\text{If } [x] = [0] \text{ then } [x]^3 + [2][x]^2 + [2][x] + [3] = [3] \neq [0]\\ &\text{If } [x] = [1] \text{ then } [x]^3 + [2][x]^2 + [2][x] + [3] = [3] \neq [0]\\ &\text{If } [x] = [2] \text{ then } [x]^3 + [2][x]^2 + [2][x] + [3] = [3] \neq [0]\\ &\text{If } [x] = [3] \text{ then } [x]^3 + [2][x]^2 + [2][x] + [3] = [4] \neq [0]\\ &\text{If } [x] = [4] \text{ then } [x]^3 + [2][x]^2 + [2][x] + [3] = [3] \neq [0]\end{aligned}$$

   It follows that we have a contradiction and no such x exists.

   QED

   (b) Show that if $a \in \mathbb{Z}$ then $a^2 \not\equiv 2 \pmod 4$ and $a^2 \not\equiv 3 \pmod 4$. [15 pts]

   Proof: We proceed by contradiction and assume that there is some $a \in \mathbb{Z}$ with either $a^2 \equiv 2 \pmod 4$ or $a^2 \equiv 3 \pmod 4$. Rewriting this with equivalence classes mod 4 we get $[a]^2 = [2]$ or $[a]^2 = [3]$. But if we check the four possibilities we see that $[0]^2 = [0]$, $[1]^2 = [1]$, $[2]^2 = [0]$ and $[3]^2 = [1]$ so no such a exists.

   QED

4. Consider the equation $6x + 15y = 33$. [20 pts]

   (a) Pass to equivalence classes for some modulus and use this to find a linear expression for x in terms of an unknown integer.

   Solution: If we pass to equivalence classes modulo 5 we get $[x] = [3]$ and so $x = 5k + 3$ for some $k \in \mathbb{Z}$.

   (b) Do the same for y using a different modulus.

   Solution: If we pass to equivalence classes modulo 2 we get $[y] = [1]$ and so $y = 2j + 1$ for some $j \in \mathbb{Z}$.

   (c) Plug your answers into the original equation to relate your two unknown integers.

   Solution: We get

$$\begin{aligned} 6(5k+3) + 15(2j+1) &= 33 \\ 30k + 18 + 30j + 15 &= 33 \\ 30k + 30j &= 0 \\ k &= -j \end{aligned}$$

   (d) Give formulas for all solutions to the equation.

   Solution: So then finally we have $x = -5j + 3$ and $y = 2j + 1$ for $j \in \mathbb{Z}$.

5. Prove that $f : (\mathbb{R} - \{1\}) \to (\mathbb{R} - \{3\})$ given by $f(x) = \frac{3x}{x-1}$ is surjective. [10 pts]

Proof: Suppose $y \in \mathbb{R} - \{3\}$. We wish to find an x so that

$$\begin{aligned} f(x) &= y \\ \frac{3x}{x-1} &= y \\ 3x &= xy - y \\ xy - 3x &= y \\ x(y-3) &= y \\ x &= \frac{y}{y-3} \end{aligned}$$

This value exists for any $y \in \mathbb{R} - \{3\}$. We must, however, check that we do not get $x = 1$ since it's not in the domain. If we did though then we'd have $1 = \frac{y}{y-3}$ or $y - 3 = y$ or $-3 = 0$, a contradiction.

QED

6. Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 6x^2 + 12x - 7$ is injective. [10 pts]

Proof: First note that $f(x) = x^3 - 6x^2 + 12x - 7 = (x-2)^3 - 1$. So now assume that $x_1, x_1 \in \mathbb{R} - \{1\}$ with $f(x_1) = f(x_2)$. Then $(x_1 - 2)^3 - 1 = (x_2 - 2)^3 - 1$ which solves easily to get $x_1 = x_2$.

QED

7. Prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 2x + 4$ is not surjective. [5 pts]

Proof: Observe that there is no $x \in \mathbb{R}$ with $f(x) = -10$ since if we try to solve:

$$\begin{aligned} x^2 - 2x + 4 &= -10 \\ x^2 - 2x + 14 &= 0 \\ x &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(14)}}{2(1)} \end{aligned}$$

which is not in $\mathbb{R}$.

$\mathcal{QED}$

8. Prove that $f : (\mathbb{R} - \{0,1\}) \to \mathbb{R}$ given by $f(x) = \frac{1}{x(x-1)}$ is not injective. [5 pts]

Proof: Observe that $3/2 \neq -\frac{1}{2}$ but $f(3/2) = f(-\frac{1}{2}) = \frac{4}{3}$.

$\mathcal{QED}$

9. Give two distinct functions $f_1, f_2 : [0,1] \to [0,1]$ which are both bijective. Prove bijectivity and distinctness. [20 pts]

Solution: Let $f_1(x) = x$ and $f_2(x) = x^2$. Clearly these are distinct since for example $f_1(0.5) \neq f_2(0.5)$.

Now then for f_1 we have:

- Surjective: For $y \in [0,1]$ we want $f_1(x) = y$ or $x = y$ so we just use $x = y$.
- Injective: If $x_1, x_2 \in [0,1]$ with $f_1(x_1) = f_1(x_2)$ we have $x_1 = x_2$.

And for f_2 we have:

- Surjective: For $y \in [0,1]$ we want $f_2(x) = y$ or $x^2 = y$ so we can use $x = \sqrt{y}$.
- Injective: For $x_1, x_2 \in [0,1]$ with $f_2(x_1) = f_2(x_2)$ we have $x_1^2 = x_2^2$ and so $x_1 = x_2$.

Note: For f_2 it's important to keep in mind that all x-values are in $[0,1]$ or else some of our calculations would not hold.