1. Consider the function : $(\mathbb{R} - \{3\}) \to (\mathbb{R} - \{1/2\})$ given by $f(x) = \frac{x}{2x-6}$.

   (a) Prove that f is injective. [5 pts]

   Proof: Suppose $x_1, x_2 \in \mathbb{R} - \{3\}$ with $f(x_1) = f(x_2)$. Then we have

$$\begin{aligned} \frac{x_1}{2x_1 - 6} &= \frac{x_2}{2x_2 - 6} \\ x_1(2x_2 - 6) &= x_2(2x_1 - 6) \\ -6x_1 &= -6x_2 \\ x_1 &= x_2 \end{aligned}$$

   QED

   (b) Prove that f is surjective. [5 pts]

   Proof: Let $y \in \mathbb{R} - \{1/2\}$. We desire an $x \in \mathbb{R} - \{3\}$ with $f(x) = y$. This means we want:

$$\begin{aligned} \frac{x}{2x - 6} &= y \\ x &= y(2x - 6) \\ x - 2yx &= -6y \\ x(2y - 1) &= 6y \\ x &= \frac{6y}{2y - 1} \end{aligned}$$

   Clearly this is defined since $y \neq 1/2$. We have to ascertain that $x \neq 3$. To do this assume it is and observe that we would get $3 = \frac{6y}{2y-1}$ which solves to get $6y - 3 = 6y$ or $-3 = 0$, a contradiction.

   QED

   (c) Find the algebraic rule for $f^{-1}(y)$. [5 pts]

   Solution: We wish to find $f^{-1}(y)$ with the property that $f(f^{-1}(y)) = y$. But this means

$$\frac{f^{-1}(y)}{2f^{-1}(y) - 6} = y$$

   Solving this yields

$$f^{-1}(y) = \frac{6y}{2y - 1}$$

   (d) Explain what you have shown about the cardinalities of $\mathbb{R} - \{3\}$ and $\mathbb{R} - \{1/2\}$. [5 pts]

   Answer: We have shown that the cardinalities are equal.

   (e) Explain non-rigorously how this idea might be extended to any two sets $\mathbb{R} - \{a\}$ and $\mathbb{R} - \{b\}$ for $a, b \in \mathbb{R}$. [10 pts]

   Answer: We can define $f : (\mathbb{R} - \{a\}) \to (\mathbb{R} - \{b\})$ by $f(x) = \frac{bx}{x-a}$ and a similar argument holds showing that these cardinalities are the same.

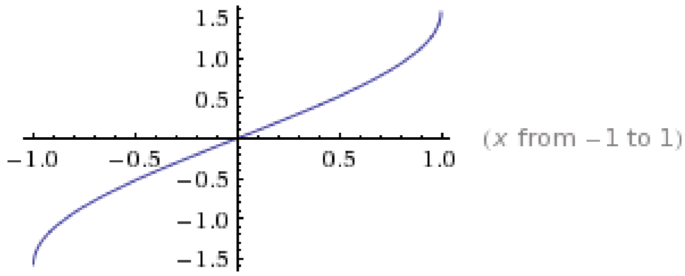
2. Consider the function $f : \mathbb{R} \to [-1, 1]$ given by $f(x) = \sin(x)$.

   (a) Does this function have an inverse? Explain. [5 pts]

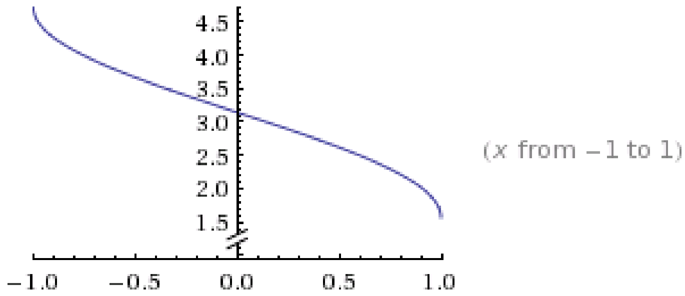
   Answer: No it does not since it is not injective. For example $f(0) = f(\pi)$.

   (b) By restricting the domain three different ways find three different inverses of f. You do not need to prove they are inverses but sketch them. [10 pts]

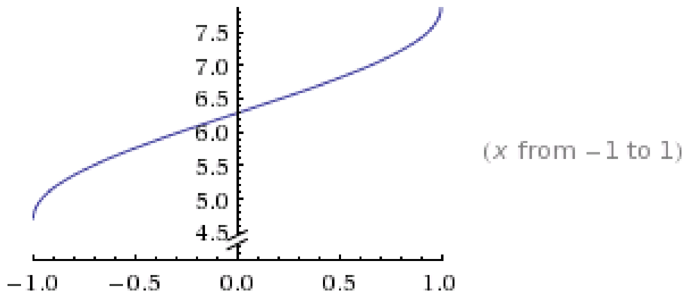
   Answer: We can restrict the domain a variety of ways. Here are three:
   $f : [-\pi/2, \pi/2] \to [-1, 1]$ for which the inverse has graph:



   $f : [\pi/2, 3\pi/2] \to [-1, 1]$ for which the inverse has graph:



   $f : [3\pi/2, 5\pi/2] \to [-1, 1]$ for which the inverse has graph:



3. Prove that if $f : A \to B$ and $g : B \to C$ are both injective then so is $g \circ f : A \to C$. [5 pts]

   Proof: Suppose $a_1, a_2 \in A$ with $(g \circ f)(a_1) = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Injectivity of g yields $f(a_1) = f(a_2)$ and then injectivity of g yields $a_1 = a_2$.

   $\mathcal{QED}$

4. Prove that $f : \mathbb{N} \to \mathbb{Z}$ given below is a bijection [10 pts]

$$f(n) = \frac{1 + (-1)^n(2n-1)}{4}$$

Proof: We have two things to show:

- Injectivity: Suppose $a, b \in \mathbb{N}$ with $f(a) = f(b)$. Then

$$\begin{aligned} \frac{1 + (-1)^a(2a-1)}{4} &= \frac{1 + (-1)^b(2b-1)}{4} \\ (-1)^a(2a-1) &= (-1)^b(2b-1) \end{aligned}$$

  Because $2a - 1$ and $2b - 1$ are both positive and the entire expressions are equal it follows that $(-1)^a$ and $(-1)^b$ must be the same sign and so we can cancel to get $2a - 1 = 2b - 1$ and so $a = b$.

  $\mathcal{QED}$

- Surjectivity: Suppose $y \in \mathbb{Z}$. We desire to find an n so that $f(n) = y$. In other words

$$\begin{aligned} \frac{1 + (-1)^n(2n-1)}{4} &= y \\ (-1)^n(2n-1) &= 4y - 1 \end{aligned}$$

  The parity of n is the only thing that can influence the sign so we must look at the parity of $4y - 1$. Note that this is either positive or negative but never 0 since $y \in \mathbb{Z}$.

  - If $4y-1 > 0$ then n must be even so we put $n = 2k$ for $k \in \mathbb{Z}$ and we get $2(2k)-1 = 4y-1$ and so $k = y$ and then $n = 2y$.
  - If $4y - 1 < 0$ then n must be odd so we put $n = 2k + 1$ for $k \in \mathbb{Z}$ and we get $-(2(2k + 1) - 1) = 4y - 1$ or $-4k - 1 = 4y - 1$ and so $k = -y$ and then $n = 2(-y) + 1 = -2y + 1$.

5. Suppose A is countably infinite. Prove that $A \times \{1, 2\}$ is also countable. [10 pts]

Note: I'll accept any reasonable explanation provided you make it clear either with a function or with a coherent and convincing listing of elements.

Proof 1: Since A is countably infinite we know A can be listed as $a_1, a_2, a_3, \ldots$. This leads to a nice listing of $A \times \{1, 2\}$ as:

$$(a_1, 1), (a_1, 2), (a_2, 1), (a_2, 2), (a_3, 1), (a_3, 2), \ldots$$

$\mathcal{QED}$

Proof 2: Alternately we can be more formal. Since A is countably infinite we know there is a bijection $f : \mathbb{N} \to A$ so then we define a function $f : \mathbb{N} \to A \times \{1, 2\}$ by

$$g(n) = \left(\lceil n/2 \rceil, \frac{3}{2} + \frac{1}{2}(-1)^n \right)$$

This function is also a bijection.

$\mathcal{QED}$

6. Prove that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$. [10 pts]

Note: I'll accept any reasonable explanation provided you make it clear either with a function or with a coherent and convincing listing of elements.

Proof: Like with the rationals (in class) we can put $|\mathbb{N} \times \mathbb{N}|$ into a grid and follow the snaking diagram to define a function $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ which is a bijection:

(1,1) (1,2) (1,3) (1,4) (1,5)
(2,1) (2,2) (2,3) (2,4) (2,5)
(3,1) (3,2) (3,3) (3,4) (3,5)
(4,1) (4,2) (4,3) (4,4) (4,5)
(5,1) (5,2) (5,3) (5,4) (5,5)