MATH 410 Exam 2 Study Guide and Worksheet

Basic Outline

4.1 D: Neighborhood.
D: Derivative as a limit and evaluation of this limit using sequences.
T: Differentiability implies continuity.
T: Differentiability of sums, products and quotients.

4.2 T: Differentiating inverses.
T: Differentiating compositions.

4.3 T: Rolle's Theorem.
T: Mean Value Theorem.
T: Identity Criterion lemma.
T: Identity Criterion.
T: Criterion for strict monotonicity.
T: Criterion for selecting maximizers and minimizers.

4.4 T: Cauchy Mean Value Theorem.
T: Deriative/approximation consequence theorem.

6.1 D: Lower and upper Darboux Sums.
D: Definition of $\underline{\int}_a^b f$ and $\overline{\int}_a^b f$.

6.2 D: Integrability.
T: Archimedes-Riemann Theorem.
D: Regular partition.
D: Gap.
T: Monotone functions are integrable.
T: Step functions are integrable.

6.3 T: Additivity, Monotonicity and Linearity of the integral.

6.4 T: Continuous functions are integrable.

6.5 T: Lemma regarding $L \leq A \leq U$ for any P.
T: First Fundamental Theorem $\int_a^b F'(x) \, dx = F(b) - F(a)$

6.6 T: Mean Value Theorem for Integrals.
T: $f : [a, b] \to \mathbb{R}$ integrable implies $\int_a^x f$ continuous.
T: Second Fundamental Theorem for $f : [a, b] \to \mathbb{R}$ continuous we have $\frac{d}{dx} \left[\int_a^x f \right] = f(x)$

How to Study

1. Definitions: Practice showing that things either do or do not satisfy definitions. This can be done either straight from the definition or via a theorem. Consider whether definitions are subsets of one another.

2. Theorems: Determine if certain specific examples fit the hypotheses for a theorem and determining the consequences. Determine if the converse of a theorem is true and if not then finding a counterexample. Determine if all hypotheses are required and if not then finding a counterexample.

3. Together: Analyzing how definitions and theorems fit with each other.

Worksheet 1
Note: Failing at the hypotheses of a theorem means failing at at least one hypothesis.

1. Let $f(x) = 3x^2 - x + 1$. Find $f'(2)$ using the definition of the derivative and using the sequence definition of the limit. Let $g(x) = |x - 2|$. Show that $g'(2)$ does not exist using the same approach.

2. For the following four one is impossible. Decide which one and give examples of the other three. For all start with x_0 and $f : I \to \mathbb{R}$ for I a neighborhood of x_0.
   Differentiable and continuous.
   Differentiable and non-continuous.
   Non-differentiable and continuous.
   Non-differentiable and non-continuous.

3. Show how the theorem on differentiating inverses can be used to find the derivative of $\sqrt{x}$.

4. Give an example demonstrating Rolle's Theorem and an example which fails at both the hypotheses and conclusion.

5. Give an example demonstrating the Mean Value Theorem and an example which fails at the hypotheses but satisfies the conclusion.

6. Give an example demonstrating the Identity Criterion and an example of two functions $f, g : D \to \mathbb{R}$ which have the same derivative but do not differ by a constant.

7. Give an example demonstrating the theorem giving a criterion for strict monotonicity, an example which fails at both the hypotheses and conclusion and an example which fails at the hypotheses but satisfies the conclusion.

8. For any positive real α give an example of a function $f : [0, 1] \to \mathbb{R}$ and a partition P such that $L(f, P) = 0$ and $U(f, P) = \alpha$.

9. Give an example of a function $f : [a, b] \to \mathbb{R}$ where $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ can be calculated explicitly from the definition.

10. For any positive real α give an example of a function $f : [0, 1] \to \mathbb{R}$ such that $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = \alpha$.

11. Give an example demonstrating the use of the Archimedes-Riemann Theorem.

12. For the following four one is impossible. Decide which one and give examples of the other three. All should be $f : [a, b] \to \mathbb{R}$.
    Continous and integrable.
    Continuous and non-integrable.
    Non-continuous and integrable.
    Non-continuous and non-integrable.

13. Give an example of $f : [a, b] \to \mathbb{R}$ which is continuous on (a, b) and non-integrable on $[a, b]$.

14. Give an example demonstrating the lemma regarding $L \leq A \leq U$ for any P and an example for which the integrability hypothesis fails and there are $A_1 \neq A_2$ satisfying the remaining hypotheses.

15. Give an example of the use of the First Fundamental Theorem.

16. Give an example demonstrating the use of the MVT4I and an example for which the hypotheses fail but the conclusion of the theorem still holds.

17. Give an example of a function $f : [0, 1] \to \mathbb{R}$ which is nonintegrable but $\int_0^x f$ is continuous on $[0, 1)$.

18. Give an example where the Second Fundamental Theorem applies but is not practical for constructing an antiderivative.

Worksheet 2

1. Assuming you wish to use sequences to work with the limit definition of a derivative for each of the following do you need to work with one specific sequence, two sequences or all sequences? Explain in each case. There may be more than one answer.

   (a) Showing $f'(x_0)$ exists.

   (b) Showing $f'(x_0)$ does not exist.

   (c) Finding $f'(x_0)$ if you know it exists.

2. Give an example of a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ exists for all x and $f^{-1}(y)$ exists for all y but $(f^{-1})'(y)$ does not exist for all y.

3. We did a homework problem in which $f : [-2, 2] \to \mathbb{R}$, $f(0) = f'(0) = 0$ and $|f''(x)| \le 4$ were given and we found a bound on $f(x)$. Explain how we can still get a bound first if $f(0)$ is a nonzero constant and then if both $f(0)$ and $f'(0)$ are nonzero constants. Hint: Define a new $g(x)$ in each case.

4. For what types of functions and partitions is it easy/practical to calculate $L(f, P)$ and $U(f, P)$? Explain.

5. Does differentiable always imply continuous?

6. Explain why it's generally impractical to calculate $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ exactly but it's often reasonable to find bounds on them. Can you give an example for this second case?

7. Suppose $f : [a, b] \to \mathbb{R}$ is given and your friend offers a proof of nonintegrability which goes as follows: He finds a sequence of partitions $\{P_n\}$ such that $\lim_{n \to \infty} U(f, P_n) - L(f, P_n) \ne 0$. Where is his mistake and why?

8. If $f : [a, b] \to \mathbb{R}$ is given list all the ways that you could try to show f to be integrable and give an example of each.

9. If we know that $f : [a, b] \to \mathbb{R}$ is integrable list all all the ways you could try to find $\int_a^b f$ and give an example of each.

10. Give an example of a function $f : [a, b] \to \mathbb{R}$ in which $\frac{d}{dx} \left[\int_a^x f \right] \ne f(x)$. Why does this not violate the Second Fundamental Theorem?

11. Explain the difference between $f : [a, b] \to \mathbb{R}$ being integrable and having an antiderivative. Could a function have one but not the other? Which? Why?