MATH 410 Sections Final Exam 1

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1. True or false: Determine if the following are true or false. If false provide a counterexample. You do not need to prove anything about the counterexamples, just provide them. [20 pts]

   (a) If a set has a supremum then it has a maximum.

   (b) If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) then it is integrable on $[a, b]$.

   (c) If $f : [a, b] \to \mathbb{R}$ is integrable on every $[c, d]$ with $a < c < d < b$ then f is integrable.

   (d) The Taylor polynomial $P_n(x)$ for a polynomial $f(x)$ of degree k equals $f(x)$ if $n \geq k$.

2. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded. This should be done by appealing to the defintions of *continuous* and *bounded*, not with a theorem. [15 pts]

3. Prove that for $b > 0$ and $n \in \mathbb{N}$ we have $\left(\frac{1}{n} + b\right)^n \geq b^n + b^{n-1}$. [15 pts]

4. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 2$. Verify the ϵ-δ criterion for continuity of $f(x)$ at $x = 3$. [15 pts]

5. Prove that $S \subseteq \mathbb{R}$ is dense iff $\forall x \in \mathbb{R}$ and $\forall \epsilon > 0$ there exists $s \in S$ within ϵ of x. [15 pts]

6. Define $f : [0, 4] \to \mathbb{R}$ by [15 pts]

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 2] \\ 2 + x & \text{if } x \in (2, 4] \end{cases}$$

   Find a formula not involving integrals for $F(x) = \int_0^x f$ and explain how you know that F is not an antiderivative of f.

7. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Use the limit definition of the derivative and the sequence definition of the limit to find $f'(2)$. Do not assume that the function is differentiable at 2. [10 pts]

8. Suppose $f, g : [0, \infty) \to \mathbb{R}$ are continuous functions such that $f(0) = g(0)$ and $f'(x) > g'(x)$ for all $x > 0$. Prove that $f(x) > g(x)$ for all $x > 0$. [15 pts]

9. Assume that $h'(1)$ exists. Prove that $\lim\limits_{x \to 1} \frac{h(x^2) - h(1)}{x - 1} = 2h'(1)$. [10 pts]

10. Use the Taylor Polynomial construction to construct a polynomial of degree 3 around $x_0 = 1$ which approximates the solution to the initial value problem $f''(x) + xf'(x) - x = f(x)$ with $f(1) = 1$ and $f'(1) = 2$. [10 pts]

11. Define $\{f_n : \mathbb{R} \to \mathbb{R}\}$ by $f_n(x) = \frac{nx}{nx^2 + 1}$. Find the function $f : [-1, 1] \to \mathbb{R}$ to which $\{f_n\}$ converges pointwise and show the convergence is not uniform. [15 pts]

12. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \sqrt{x + 1}$. Using our proof of the Weierstrass Approximation Theorem which degree polynomial would suffice for approximating $f(x)$ to within $\epsilon = 0.1$? Write down this polynomial in Σ form. [15 pts]

13. Use a Taylor polynomial to calculate $\sin(0.5)$ to within 0.01. [15 pts]

14. Define $f_n : [-1, 1] \to \mathbb{R}$ by $f_n(x) = \sum\limits_{k=1}^{n} \frac{x^k}{k^2 3^k}$. Show that $\{f_n\}$ converges uniformly. [15 pts]